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ON FUNCTIONS OF SEVERAL COMPLEX VARIABLES*

BY

WILLIAM F. OSGOOD

By means of certain very simple examples it is possible to decide a number of questions relating to analytic functions of several complex variables, which have hitherto, so far as the author is aware, remained open.

It is unsatisfactory, in stating an important theorem, not to know whether a given hypothesis is needed merely for convenience of proof, or whether the theorem would be false if it were omitted. The situation is still more annoying when it is conceivable that the theorem could be proven with about the same ease without the hypothesis, if one were only able to see more clearly.

Some of the questions to which these examples apply are of this nature. In other cases analytic phenomena of novel character are brought to light.

1. A BRANCH MORE GENERAL THAN AN ELEMENT

Let x, y, and z be defined by the equations:

$$x = f(u, v), \qquad y = \phi(u, v), \qquad z = \psi(u, v), \tag{1}$$

where f, ϕ, ψ are rational functions of u and v, all of which vanish at the origin, $(u, v) = (0, 0)$; and let at least one of the two-rowed determinants in the matrix

$$\left\| \begin{matrix} \dfrac{\partial f}{\partial u} & \dfrac{\partial \phi}{\partial u} & \dfrac{\partial \psi}{\partial u} \\ \dfrac{\partial f}{\partial v} & \dfrac{\partial \phi}{\partial v} & \dfrac{\partial \psi}{\partial v} \end{matrix} \right\| \tag{2}$$

be not identically 0. Then the equations (1) determine a piece of a monogenic analytic configuration—a surface—and the equation of this surface can be obtained in the form

$$G(x, y, z) = 0 \tag{3}$$

by eliminating u and v. Here, G denotes an irreducible polynomial vanishing at the origin.

If we replace the requirement that f, ϕ, ψ be rational by the hypothesis

* Presented to the Society, October 30, 1915.

that they be merely analytic at the origin, and retain the other requirements, the equations (1) still determine a piece of a monogenic analytic configuration, and it is frequently possible to represent this piece in the neighborhood of the origin by a single equation (3), where now G is analytic at the origin and vanishes there, and moreover is irreducible.* The question presents itself: Is this always possible?

This question is to be answered in the negative, as is shown by the following example.

THEOREM I. *The equations*

$$(A) \qquad x = u, \qquad y = uv, \qquad z = uve^v$$

define a piece of a monogenic analytic configuration G, the coördinates of whose points do not satisfy any relation of the form

$$\Omega(x, y, z) = 0,$$

where Ω is analytic at the origin, but does not vanish identically.

The proof is immediate. Suppose there were such a function. Let it be developed into a series of homogeneous polynomials:

$$\Omega = \Omega_0 + \Omega_1 + \Omega_2 + \cdots,$$

and let

$$\Omega_n = A_0 x^n + (A_1 y + B_1 z) x^{n-1} + (A_2 y^2 + B_2 yz + C_2 z^2) x^{n-2} + \cdots.$$

Substitute for x, y, z in Ω their values from (A) and develop according to ascending powers of u. The coefficient of u^n will be

$$(4) \qquad A_0 + (A_1 + B_1 e^v) v + (A_2 + B_2 e^v + C_2 e^{2v}) v^2 + \cdots,$$

and this coefficient must vanish for all values of v. Let $v = v_1$ be an arbitrary value of v. Then, in particular, (4) must vanish for the values of the argument,

$$(5) \qquad v = v_1 + 2k\pi i, \qquad k = 0, 1, 2, \cdots, n.$$

The equations thus arising from (4) may be regarded as $n + 1$ linear homogeneous equations in A_0 and the parentheses. Their determinant, being the product of the differences of the numbers (5), is not 0. Hence A_0 and each parenthesis must vanish.

Finally, if an expression of the form

$$A + Be^v + Ce^{2v} + \cdots \text{ to } k \text{ terms}$$

vanishes for all values of v, then each coefficient must vanish. For, on setting

$$e^v = w,$$

* I. e., G cannot be represented as the product of two factors, each of which is analytic at the origin and vanishes there.

this expression becomes a polynomial,

$$A + Bw + Cw^2 + \cdots .$$

This completes the proof. The example can be carried over to n functions of $n - 1$ parameters by setting

$$(A') \qquad \begin{gathered} x_1 = u_1, \qquad x_2 = u_1 u_2, \qquad x_3 = u_1 u_2 e^{u_2}, \\ x_k = u_1 u_2 e^{u_{k-1}}, \qquad k = 4, \cdots, n. \end{gathered}$$

The title of this paragraph is due to the fact that Weierstrass* suggests by a distinction in terminology that he was aware of the phenomena here treated. Such equations as (1) define what he calls a *branch* of a monogenic analytic configuration, while an *element* is given by a more restricted definition.

2. On the identical vanishing of the jacobian

The following theorem is well known.† Let

$$(B) \qquad f_i(u_1, \cdots, u_n, y_1, \cdots, y_m) \qquad (i = 1, \cdots, n),$$

be analytic in the point $(b, \beta) = (b_1, \cdots, b_n, \beta_1, \cdots, \beta_m)$, and let

$$f_i(b_1, \cdots, b_n, \beta_1, \cdots, \beta_m) = a_i \qquad (i = 1, \cdots, n).$$

Let $\mathfrak{T}$ be an arbitrary neighborhood of the point

$$(a, \beta) = (a_1, \cdots, a_n, \beta_1, \cdots, \beta_m).$$

Finally, let the jacobian

$$\frac{\partial(f_1, \cdots, f_n)}{\partial(u_1, \cdots, u_n)},$$

regarded as a function of the $n + m$ arguments (u, y), vanish identically.

Then there exists a point (a', β') of $\mathfrak{T}$ and a function

$$\Omega(x_1, \cdots, x_n, y_1, \cdots, y_m)$$

analytic at (a', β') and such that

$$(i) \qquad \Omega(x_1, \cdots, x_n, \beta_1', \cdots, \beta_m') \not\equiv 0;$$

if x_i be replaced by f_i:

$$(ii) \qquad x_i = f_i(u_1, \cdots, u_n, y_1, \cdots, y_m) \qquad (i = 1, \cdots, n),$$

the function Ω then goes over into a function of $(u_1, \cdots, u_n, y_1, \cdots, y_m)$

* Weierstrass, in the memoir devoted to the proof of the Theta Theorem, *Werke*, vol. 3, p. 103, where the definition of *Zweig* is given, the definition of *Element* having just preceded.

† Cf. Peano-Genocchi, *Calcolo differenziale*, p. 172; Jordan, *Cours d'analyse*, vol. I, 2d ed., 1893, p. 86.

which vanishes identically in these arguments:

$$\Omega(f_1, \cdots, f_n, y_1, \cdots, y_m) \equiv 0.$$

Is this theorem true when the point (a', β') is taken at (a, β)? The answer is given by the following theorem.

THEOREM 2. *a) In the case $n = 2$ the above theorem is true when the point (a', β') is taken at (a, β).*

b) When $n > 2$, the theorem does not hold generally when (a', β') is taken at (a, β).

The first part of the theorem is due to Bliss, and was proven by him in the *Princeton Colloquium* of 1909, published in 1913, Chap. 2, § 12, p. 67.

To prove the second part of the theorem it is sufficient to cite one example. In the equations (A) let u be replaced by uw:

$$(B) \qquad x = uw, \qquad y = uvw, \qquad z = uvwe^{v}.$$

Here, it is clear that the jacobian vanishes identically, and that, moreover, the three functions are not connected by any relation of the form

$$\Omega(x, y, z) = 0,$$

where Ω is analytic at the origin and does not vanish identically, but goes over, when the functions (B) are substituted in it, into a function of u, v, w which is identically zero.

For the case of n arguments u_i and m parameters y_k we may set

$$(B') \qquad \begin{aligned} &x_1 = u_1 u_n, \qquad x_2 = u_1 u_2 u_n, \qquad x_3 = u_1 u_2 u_n e^{u_2}, \\ &x_k = u_1 u_2 u_n e^{u_{k-1}} \qquad (k = 4, \cdots, n). \end{aligned}$$

3. CONCERNING A GENERALIZATION OF WEIERSTRASS'S THEOREM OF FACTORIZATION

In the *Madison Colloquium** I called attention to a generalization of Weierstrass's theorem of factorization for functions of two variables, and raised the question as to whether the theorem holds generally.

Let $F(u; x_1, \cdots, x_n)$ be analytic at the origin and vanish there. *Is it then always possible to represent F in the neighborhood of the origin in the form*

$$F(u; x_1, \cdots, x_n) = [A_0 u^m + A_1 u^{m-1} + \cdots + A_m] H(u; x_1, \cdots, x_n),$$

where $A_k(x_1, \cdots, x_n)$, $k = 0, \cdots, m$, is analytic at the origin, and $H(u; x_1, \cdots, x_n)$ is analytic at the origin and does not vanish there?

That the theorem is true when $n = 1$ was recognized by Risley and Macdonald.† When, however, $n > 1$, the theorem is false, as appears from the following example.

* P. 184.

† Annals of Mathematics, 2d ser., vol. 12 (1910), p. 73.

Let $w = Q(z)$ be the function defined by the series*

$$Q(z) = z + \sum_{n=2}^{\infty} \frac{z^{n!+2}}{(n!+1)(n!+2)}.$$

This function is analytic throughout the interior of the unit circle, has the circumference of that circle as a natural boundary, and maps the above region in a one-to-one manner and conformally on a finite single-leaved region of the w-plane. Let $R(w)$ be the inverse function:

$$w = Q(z), \qquad z = R(w).$$

Then R is also single-valued and analytic, the point $w = 0$ lying within its domain of definition.

We proceed now to form the example required for our purpose. Consider the function

$$(C) \qquad z = Q\left(\frac{y}{x}\right).$$

It is obviously a single-valued monogenic analytic function, and its domain of definition is the region

$$(6) \qquad 0 \leqq |y| < |x|.$$

Moreover, z remains finite: $|z| < G$.

The corresponding monogenic analytic configuration is also given by the equation

$$(D) \qquad y - xR(z) = 0.$$

The example is the following:

$$(E) \qquad F(z, x, y) = y - xR(z).$$

If this function permitted a factorization of the kind in question, then the configuration (D) would be represented in the neighborhood of the origin by an equation of the form

$$(7) \qquad A_0(x, y)z^m + A_1(x, y)z^{m-1} + \cdots + A_m(x, y) = 0,$$

where $A_k(x, y)$ is analytic at the origin, and $A_0(x, y) \not\equiv 0$.

Here, $A_0(x, y)$ must vanish at the origin. For otherwise we may set $A_0(x, y) = 1$, and then either horn of the dilemma: $m > 1$, $m = 1$, leads at once to a contradiction.

Now let (x_1, y_1) be chosen within an arbitrary neighborhood of the origin as a point distinct from the origin, for which $A_0(x, y)$ vanishes, but for which some other coefficient, $A_k(x, y)$, does not. It must be possible to find such a point, since otherwise all the coefficients in (7) would be divisible

* Cf. the author's *Funktionentheorie*, vol. 1, ch. 9, § 5.

by one and the same prime factor $G(x, y)$ of $A_0(x, y)$; hence (E) must also be divisible by G, and this is evidently not the case.

Let (x_0, y_0) be a point of the domain (6), lying in the neighborhood of the origin, and let $A_0(x_0, y_0) \neq 0$. Let the discriminant $D(x, y)$ of (7) also be different from 0 in this point. Then $z_0 = Q(y_0/x_0)$ will be one of the roots of (7).

We now proceed to let the point (x, y) move along a path emanating from (x_0, y_0) and leading to (x_1, y_1), but passing through no point in which either $A_0(x, y)$ or $D(x, y)$ vanishes. The function z defined in the neighborhood of (x_0, y_0) by (C) must admit an analytic continuation along this path, since this is true for the same function, regarded as a root of (7).

On the other hand, at least one of the roots of (7) becomes infinite as (x, y) approaches (x_1, y_1), and we can choose the path so that the above function z will be carried over into such a root. For, even though (7) may be reducible, still the function (C) satisfies an irreducible equation of the same type as (7), the left-hand side of the equation being a factor of the left-hand side of (7); and we may choose (x_1, y_1) with reference to the irreducible equation. Here is a contradiction, and thus the following theorem is established.

THEOREM 3. *When $n = 1$, the question proposed at the beginning of the paragraph is to be answered in the affirmative; when $n > 1$, in the negative.*

A further example which illustrates the same point, but in which the whole monogenic analytic configuration lies in the finite region is the following:

$$(F) \qquad x = u, \qquad y = uv, \qquad z = Q(u) + Q(v), \qquad |u| < 1, \qquad |v| < 1;$$

or:

$$(F') \qquad z = Q(x) + Q\left(\frac{y}{x}\right), \qquad 0 \leqq |y| < |x| < 1;$$

or still again:

$$(F'') \qquad y = xR[z - Q(x)].$$

In the last case, $|x| < 1$, $Q(x)$ remains finite, and z must be so chosen that the point $z - Q(x)$ lies in the domain of definition of the function R. Thus y also remains finite.

4. GEOMETRIC INTERPRETATION. A FURTHER EXAMPLE

Consider the function

$$(G) \qquad z = \frac{y}{x}$$

in the neighborhood of the origin, $(x, y) = (0, 0)$. The function is analytic at all points there save those for which $x = 0$. But its value does not remain finite.

In fact, we can picture the function to ourselves in a way by considering

the real surface corresponding to real values of x and y. This surface is spread out over the neighborhood of the origin, the line $x = 0$ in the (x, y)-plane being excepted, and is generated by a line

$$(g) \qquad y = z_0 x, \qquad z = z_0,$$

which always cuts the z-axis, is parallel to the (x, y)-plane, and rotates as its point of intersection with the z-axis advances. The one direction in the (x, y)-plane to which this line never becomes parallel is that of the y-axis; and in order to make a small angle with that direction, it has to recede far.

On the other hand, consider the function (C):

$$(H) \qquad z = Q\left(\frac{y}{x}\right),$$

and set over against it the surface

$$(H') \qquad y = xR(z).$$

Think of this surface as generated by the moving line:

$$(h) \qquad y = R(z_0)x, \qquad z = z_0.$$

This is a surface of precisely the same character as the one generated by the moving line (g), the difference being merely that the rate at which the moving line rotates as its point of intersection with the z-axis travels along that line is different; and, moreover, that this point does not go so far, being restricted to a finite field.

When we project the surface (H') on the (x, y)-plane, we do not get approximately the whole neighborhood of the origin, but only so much, in the domain of reals, as lies within a certain angle.

This fact is suggestive for the complex case. The analogue of the angular region just referred to is the portion of the four-dimensional space defined by the relations (6).

That which the two examples (G) and (H) have in common is this, that when we consider the part of the monogenic analytic configuration which lies in the (complex) three-dimensional neighborhood of the origin $(x, y, z) = (0, 0, 0)$, we obtain each time a piece of an analytic surface,

$$(K) \qquad y = \omega(x, z),$$

where ω is analytic at the origin and vanishes there, namely

$$\omega(x, z) = xz \qquad \text{resp.} \qquad \omega(x, z) = xR(z).$$

That in which the two examples differ is this. When the equation (K) is solved for z, we get in the one case a function (G) having a non-essential singularity at the origin, $(x, y) = (0, 0)$. In the other case, the solution

of (K) for z yields a function (H) having at the origin a singularity for which the algebraic functions yield no prototype. The origin lies on the boundary of a lacunary space; it is a point of a natural boundary of a single-valued monogenic function of x and y.

Thus we have been led by the geometric picture to an example which justifies the following theorem.

THEOREM 4. *Let $G(x, y, z)$ be a function analytic at the origin and vanishing there, and let G be irreducible. Furthermore, let G actually depend on z. Then the equation $G(x, y, z) = 0$, when solved for z may yield a single-valued monogenic analytic function which has at the origin $(x, y) = (0, 0)$ a singularity for which algebraic and algebroid functions yield no prototype.*

In fact, the origin may be a point of a natural boundary, the neighborhood of the origin consisting in part of a lacunary space.

Two remarks.—First, we have obtained a property of analytic functions in two variables which is distinctly unlike what we know to be the case for such functions in a single variable. There, if $y = f(x)$ is a function analytic at a point $x = a$ or having there at most a branch-point or pole (the most general singularity of which algebraic or algebroid functions of one variable are capable), and if $f(x)$ actually depends on x (i. e., is not a constant) then the inverse function $x = \phi(y)$ can have at the corresponding point at most a singularity of algebraic type.

The second remark is this. While it is easy to form real functions of two real variables which are analytic in disconnected regions of the (x, y)-plane, but are, in the complex domain, merely branches of one and the same monogenic analytic function,—log sin $(x + y)$ is, for example, such a function,—still, the monogenic functions involved are apt to be multiple-valued.

In the functions (F') and (H) we have *single-valued* monogenic analytic functions exhibiting this property.

If it be desired to have the two domains of definition finite and without even a boundary point in common, the following function will serve the purpose:

$$(L) \qquad F(x, y) = Q(x) + Q\left(\frac{1}{2x}\right) + Q\left(\frac{y}{x}\right),$$

$$\tfrac{1}{2} < |x| < 1, \qquad 0 \leqq |y| < |x|.$$

HARVARD UNIVERSITY,
June 10, 1915

A STUDY OF CERTAIN FUNCTIONAL EQUATIONS FOR THE ϑ-FUNCTIONS*

BY

E. B. VAN VLECK AND F. H'DOUBLER

Functional equations for familiar elementary functions have been studied by Cauchy and others. In the case of the sine and cosine it is natural to treat the two functions together and to take as the defining system the addition-equations,

$$(D)\qquad \begin{aligned} S(x+y) &= S(x)C(y) + C(x)S(y), \\ C(x+y) &= C(x)C(y) - S(x)S(y). \end{aligned}$$

This system of equations has been studied by Tannery† and in Osgood's *Lehrbuch der Funktionentheorie*,‡ and the real solutions are shown to have the form

$$S(x) = e^{ax}\sin bx, \qquad C(x) = e^{ax}\cos bx,$$

from which the trigonometric functions are separated by imposing the accessory condition

$$S^2(x) + C^2(x) = 1.$$

The general solution of (D) is given incidentally in this paper (§ 7) on the hypothesis that $S(x)$, $C(x)$ are continuous functions (not necessarily analytic) existent over the entire complex x-plane.

So far as we know, no previous investigation has been made of the more comprehensive system

$$(A)\qquad \begin{aligned} \phi(x+y)\phi(x-y) &= \phi^2(x)\psi^2(y) - \phi^2(y)\psi^2(x), \\ \psi(x+y)\psi(x-y) &= \psi^2(x)\psi^2(y) - \phi^2(x)\phi^2(y), \end{aligned}$$

which forms the object of study of this paper. Our purpose is two-fold, to ascertain what solutions this system admits and, secondly, when $\phi(x)$, $\psi(x)$ possess doubly periodic systems of zeros, to develop directly from (A) the characteristic properties found for them in the theory of the Jacobian

* Presented to the Society at its summer meeting in Madison, September 9, 1913.

† *Fonctions d'une variable*, 1886, p. 147.

‡ See second edition, p. 582.

ϑ-functions or the Weierstrassian σ-functions. In connection with (A) the functional equation

$$(B) \qquad R(x+y)R(x-y) = \frac{R^2(x) - R^2(y)}{1 - R^2(x)R^2(y)}$$

is also treated.

It will be supposed that $\phi(x)$, $\psi(x)$ are continuous one-valued functions (not necessarily analytic) existent over the entire plane, except in the investigation of real solutions of (A) when existence is only assumed for the real axis. In any solution the components $\phi(x)$, $\psi(x)$ can be modified by a common arbitrary exponential factor of form (11).

The different kinds of solutions can be differentiated from one another by the manner in which the roots are distributed. If $\phi(x) \equiv 0$, we have the equation

$$(C) \qquad \psi(x+y)\psi(x-y) = \psi^2(x)\psi^2(y),$$

of which the complete solution is obtained (§ 2). A second case is that in which $\phi(x)$ vanishes identically along a straight line through the origin, or over a set of equidistant parallel lines. This case is discharged in § 3. The solutions are non-analytic and are deducible in most simple fashion from sections of solutions for the other cases taken along the real axis or other line through the origin.

The only other possible cases are those in which the zeros of $\phi(x)$ and those of $\psi(x)$ form simultaneously either simply periodic or doubly periodic systems. A necessary and sufficient condition for the simply periodic case is obtained, depending upon the value of $\phi(x)/\phi(\alpha/2)$ for $x = \alpha/4$, where α designates a "primitive" zero used in normalizing $\phi(x)$. The complete solution (§ 7) is obtained for this case, the analytic solutions being expressible through the functions

$$\phi(x) = e^{cx^2} \sin \mu x, \qquad \psi(x) = e^{cx^2} \cos \mu x,$$

or through "equivalent" (§ 1) solutions. All non-analytic solutions of this case are obtainable from these by a real affine transformation of the x-plane which leaves the origin unaltered. It may be added that by simple transformations equations (A) may be reduced in this simply periodic case to form (D).

In the remaining case of a doubly periodic system of zeros the existence of three "allied solutions" (§ 6) is established which together involve four functions having the characteristic properties of the four ϑ-functions. When analytic, the solution is expressible through

$$\phi(x) = \frac{\vartheta_1(cx, q)}{\vartheta(0, q)} e^{c'x^2}, \qquad \psi(x) = \frac{\vartheta(cx, q)}{\vartheta(0, q)} e^{c'x^2},$$

and equivalent solutions, or through other similar pairs of ϑ-functions (cf.

I–III). It seems altogether likely that all non-analytic solutions are derivable from these by affine transformation of the x-plane, but the point is not settled.

All solutions are analytic on any line through the origin. A complete determination of the real solutions (§ 10) is made, three cases (beside transitional cases) being distinguished which are solved through the functions:

$$\text{(I)} \qquad \phi_1(x) = \frac{\vartheta_1(cx, q)}{\vartheta(0, q)} e^{c'x^2}, \qquad \psi(x) = \pm \frac{\vartheta(cx, q)}{\vartheta(0, q)} e^{c'x^2},$$

$$\text{(II)} \qquad \phi_2(x) = \frac{\vartheta_1(cx, q)}{\vartheta_2(0, q)} e^{c'x^2}, \qquad \psi(x) = \pm \frac{\vartheta_2(cx, q)}{\vartheta_2(0, q)} e^{c'x^2},$$

$$\text{(III)} \qquad \phi_3(x) = \frac{\vartheta_1(cx, iq)}{\vartheta_2(0, iq)} e^{c'x^2}, \qquad \psi(x) = \pm \frac{\vartheta_2(cx, iq)}{\vartheta_2(0, iq)} e^{c'x^2},$$

in which q, c, c' are real and arbitrary with $|q| < 1$. In terms of Weierstrassian functions we can give the solutions the more symmetrical form

$$\phi(x) = e^{c'x^2} \sqrt{(e_\alpha - e_\beta)(e_\alpha - e_\gamma)}\, \sigma(cx), \qquad \psi(x) = \pm e^{c'x^2} \sigma_\alpha(cx) \quad (\alpha = 1, 2, 3).$$

For $\alpha = 1, 3$ the quasi-periods of the functions are the one real and the other a pure imaginary, and we have the order of succession $e_1 > e_2 > e_3$. When $\alpha = 2$, the periods are conjugate imaginaries, e_2 is real, and e_1, e_3 are conjugate imaginaries. In the transition case we get the solutions:

$$\text{(IV)} \qquad \phi(x) = e^{cx^2} \sinh \mu x, \qquad \psi(x) = \pm e^{cx^2} \cosh \mu x,$$

$$\text{(V)} \qquad \phi(x) = e^{cx^2} \sin \mu x, \qquad \psi(x) = \pm e^{cx^2} \cos \mu x.$$

It should be mentioned that it follows directly from (A) that $\phi(x)$ satisfies the *équation à trois termes:*

$$f(u_1 + u_2)f(u_1 - u_2)f(u_3 + u_4)f(u_3 - u_4) + f(u_1 + u_3)f(u_1 - u_3)f(u_4 + u_2)f(u_4 - u_2) + f(u_1 + u_4)f(u_1 - u_4)f(u_2 + u_3)f(u_2 - u_3) = 0.$$

The solution of this equation was obtained by Delisle* through power-series methods on the hypothesis of the analytic character of $f(u)$. With one assumption confessedly unproved regarding a certain determinant, the solution was shown to be of form $c'\, e^{cx^2} \sigma(x)$.

1. On the zeros of the solutions

Let $\phi(x)$, $\psi(x)$ be one-valued, continuous functions satisfying the equations

*Mathematische Annalen, vol. 30 (1887), p. 91.

$$(A)\qquad \begin{cases} (1)\quad \phi(x+y)\phi(x-y) = \phi^2(x)\psi^2(y) - \phi^2(y)\psi^2(x), \\ (2)\quad \psi(x+y)\psi(x-y) = \psi^2(x)\psi^2(y) - \phi^2(x)\phi^2(y). \end{cases}$$

We shall suppose their existence over the whole of the finite complex plane, except in parts of the work in which their existence is assumed only on a line through the origin, taken without loss of generality as the real axis. Much of the development will be seen to be applicable on either hypothesis.

From the exchange of x and y in (A) it appears that

$$\phi(x-y) = -\phi(y-x), \qquad \psi(x-y) = \psi(y-x);$$

in other words, $\phi(x)$ is an odd and $\psi(x)$ an even function. Since $\phi(0) = 0$, we obtain by putting $y = 0$ in (2) either $\psi^2(0) = 1$ or $\psi(x) \equiv 0$. If $\psi(x) \equiv 0$, it follows from (2) that $\phi(x) \equiv 0$, and this is also true if

$$\psi(x) \equiv \pm 1.$$

We put to one side these trivial solutions, which are, indeed, the only ones in which both functions are constants.

It is immaterial whether we take $\psi(0) = 1$ or $\psi(0) = -1$ since the four pairs of functions $\pm\phi(x)$, $\pm\psi(x)$ simultaneously satisfy (A). We take henceforth $\psi(0) = 1$. Other pairs of functions satisfying (A) are $\pm i\phi(x)$, $\pm\psi(x)$. These eight solutions will be called *equivalent.*

The key to the properties of the two functions is to be found in the distribution of their zeros. We will first show that $\phi(x)$ *and* $\psi(x)$ *can not vanish simultaneously for any value of* x. Suppose, if possible, that they both vanish for $x = \xi$. Then on placing $y = x + \xi$ equations (A) become

$$(3)\qquad \begin{aligned} 0 &= \phi^2(x)\psi^2(x+\xi) - \phi^2(x+\xi)\psi^2(x), \\ 0 &= \psi^2(x)\psi^2(x+\xi) - \phi^2(x+\xi)\phi^2(x). \end{aligned}$$

Since $\psi(x)$ is by hypothesis continuous and takes the value 1 at the origin, we can choose a positive δ so small that $\psi(x)$ will not vanish in the circle (interval) $|x| \leqq 2\delta$. If either $\phi(x+\xi)$ or $\psi(x+\xi)$ vanishes for a value of x in this circle (interval), equations (3) show that the other does also. When neither vanishes, their elimination from (3) gives $\phi^4(x) = \psi^4(x)$. Now for $y = x$ equation (2) yields the fundamental relation

$$(4)\qquad \psi(2x) = \psi^4(x) - \phi^4(x),$$

from which we get $\psi(2x) = 0$. This, however, is impossible for $|x| \leqq \delta$. It follows therefore that if $\phi(x)$ and $\psi(x)$ both vanish at any point ξ, they must vanish over a circle (interval) of radius δ having ξ as its center. Follow now our functions along a straight line from ξ to the origin. The interval in which both functions simultaneously vanish can not terminate at a point ξ'

between ξ and 0 because then both functions in consequence of their continuity would vanish at ξ' and hence in some circle (interval) having ξ' as its center. On the other hand, the interval can not extend to the origin because $\psi(0) = 1$. From the resulting contradiction we conclude that the two functions can nowhere vanish at the same point.

We will next establish that when $\phi(x), \psi(x)$ exist over the complex plane, *one or the other must have a zero not at the origin.* For suppose the contrary and consider the quotient $R(x) = \phi(x)/\psi(x)$ which satisfies the functional equation

$$(B) \qquad R(x+y)R(x-y) = \frac{R^2(x) - R^2(y)}{1 - R^2(x)R^2(y)}.$$

By the hypothesis just made $R(x)$ is both finite and continuous, vanishing only for $x = 0$. The right-hand member of (B) is indeterminate only if $R^2(x) = R^2(y) = \pm 1$. It is, however, impossible for $R^2(x)$ to take the values ± 1, for we would then have $\phi^4(x) = \psi^4(x)$ and hence also $\psi(2x) = 0$, contrary to the hypothesis. Furthermore, $R(x)$ can not take the *same* value at two different points x, y because then by (B) we would have $R(x+y) = 0$, i. e., $y = -x$, which is impossible since then $R(y) = R(-x) = -R(x)$. Consequently $R(x)$ must build the complex x-plane by a continuous 1–1 correspondence upon a portion of the R-plane.

In this correspondence a simply connected region including the origin of the x-plane is built upon a simply connected region including the origin of the R-plane. To the continuous set of circles of increasing radius about the x-origin corresponds a succession of closed contours about the origin of the R-plane which fill an open region T which is the image of the finite x-plane. If T does not extend infinitely far outwards from the origin O in every direction save in the directions of the points $x = \pm 1, \pm i$, there must be in some other direction a first frontier point $\overline{R}$. On $O\overline{R}$ we can select points $R(x)$, $R(y)$ so close to $\overline{R}$ that the corresponding points x, y lie on circles of an arbitrarily large radius about the origin, and we can at the same time take $|y|$ so large relatively to $|x|$ that $|x+y|$ and $|x-y|$ will both be as large as we please. On the other hand, when $R(x)$ and $R(y)$ are sufficiently near to $\overline{R}$ and therefore to each other, it follows from (B) that either $|R(x+y)|$ or $|R(x-y)|$, and hence either $|x+y|$ or $|x-y|$, must be extremely small. Since this gives a contradiction, the frontier point $\overline{R}$ can not exist. Then T must stretch out to infinity in every direction from the origin except toward $\pm 1, \pm i$. Consequently within T we can find points for which $R(x)R(y) = \pm 1$, whence it follows by (B) that either $R(x+y)$ or $R(x-y)$ must be infinite. But this is impossible if always $\psi(x) \neq 0$. Thus finally we must reject the hypothesis that neither $\phi(x)$ nor $\psi(x)$ vanishes excepting the former at $x = 0$.

Suppose now $\alpha \neq 0$ to be a root of $\phi(x)$. For $y = x + \alpha$ equation (1) gives

$$\frac{\phi^2(x)}{\psi^2(x)} = \frac{\phi^2(x+\alpha)}{\psi^2(x+\alpha)}. \tag{5}$$

Consequently α is a period of $\phi^2(x)/\psi^2(x)$, and $\phi(x)$ which vanishes for $x = 0$ has the periodic set of zeros $\pm n\alpha$. If $\phi(x)$ is analytic, its complete system of zeros is simply or doubly periodic unless $\phi(x)$ vanishes over the entire plane. When $\phi(x)$ is merely supposed to be continuous, the same conclusion follows by a well-known argument* provided there is a finite lower limit for the absolute values of all zeros other than $\alpha = 0$. If there is no such lower limit, then either the zeros for sufficiently small value of their modulus will all have the same argument (mod. π) or we can form doubly periodic systems of zeros, for which the sides of the period parallelogram are smaller than any arbitrarily assigned magnitude. In the latter case the zeros will be everywhere dense and $\phi(x)$ will vanish over the entire plane. In the former case there is a line through the origin on which there exists arbitrarily small periods for the zeros and on which accordingly $\phi(x)$ everywhere vanishes. If, furthermore, $\phi(x)$ has a zero without this line, then from the periodicity of $\phi^2(x)/\psi^2(x)$ it follows that there is a set of equidistant parallel lines on which $\phi(x)$ everywhere vanishes.

Suppose next that $\psi(x)$ has a root β. For $y = x + \beta$ equation (2) becomes

$$\frac{\phi^2(x)}{\psi^2(x)} = \frac{\psi^2(x+\beta)}{\phi^2(x+\beta)}. \tag{6}$$

Consequently $\phi^2(x)/\psi^2(x)$ has the period 2β, and 2β must be a zero of $\phi(x)$. Since also either $\phi(x)$ or $\psi(x)$ possesses a root not at the origin, we conclude that $\phi(x)$ *has always one such zero and therefore a periodic system of zeros.* Furthermore, from (6) it is evident that the zeros of either function are obtained from those of the other by a displacement β. Thus *if* $\psi(x)$ *has a single zero, the system of zeros of* $\phi(x)$ *and* $\psi(x)$ *must be of the same character.* Later it will be shown (§ 5) that if the zeros of $\phi(x)$ are isolated and either simply or doubly periodic, $\psi(x)$ will possess zeros also.

According to the nature of the root system of $\phi(x)$ the following four cases may now be distinguished, and only these:

(1) $\phi(x) \equiv 0$.

(2) $\phi(x)$ vanishes everywhere upon some straight line through the origin or over a set of equidistant parallel straight lines, one of which passes through $x = 0$.

(3) $\phi(x)$ and $\psi(x)$ have each a simply periodic set of zeros, those of the

* Cf. Osgood's *Funktionentheorie*, 2d ed., pp. 459–460.

latter function being situated half way between consecutive zeros of the former, inasmuch as 2β is a root of $\phi(x)$.

(4) Both functions have a doubly periodic set of zeros. Three positions are possible for the zeros of $\psi(x)$ relatively to those of $\phi(x)$; they may be either the centers of the period-parallelograms for the zeros of $\phi(x)$, or the middle points of one system of parallel sides, or the middle points of the other.

2. A SPECIAL FUNCTIONAL EQUATION

We shall consider first the case in which $\phi(x)$ vanishes over the entire complex plane. Equation (2) then reduces to the functional equation

$$\psi(x+y)\psi(x-y) = \psi^2(x)\psi^2(y). \tag{C}$$

More generally, this equation holds for any line through the origin on which $\phi(x)$ vanishes identically.

Consider now any such line and assume for $\psi(x)$ an arbitrary value at some initial point $x = x'$ of the line. Unless $\psi(x)$ vanishes over the entire line, we can take a point for which the initial value is not zero. Seek now to determine the form of $\psi(x)$ on the line.

For $y = x$ we have

$$\psi(2x) = \psi^4(x) \tag{7}$$

since we may take $\psi(0) = 1$, precisely as in § 1.

The formula

$$\psi(nx) = [\psi(x)]^{n^2} \tag{8}$$

can now be established for positive integral values of n by mathematical induction. For assume it to hold up to a particular value of n. When $y = nx$, equation (C) gives

$$\psi[(n+1)x] = \frac{\psi^2(nx)\psi^2(x)}{\psi[(n-1)x]} = \frac{[\psi(x)]^{2n^2+2}}{[\psi(x)]^{(n-1)^2}} = [\psi(x)]^{(n+1)^2}.$$

Hence (8) continues to hold when n is replaced by $n+1$.

From (8) we know $\psi(x)$ over the point set $\{nx'\}$, $(n = 2, 3, \cdots)$. Let x in (7) be now replaced by $x'/2$. We get

$$\psi\left(\frac{x'}{2}\right) = [\psi(x')]^{1/4}, \tag{9}$$

whence by repeatedly halving the argument we obtain

$$\psi\left(\frac{x'}{2^m}\right) = [\psi(x')]^{1/2^{2m}}. \qquad (m = 1, 2, \cdots). \tag{10}$$

In the successive applications of (9) to halve the argument any succession of

fourth roots can be chosen at the start, but ultimately the choice is restricted by the requirement that as $x'/2^m$ approaches the origin, $\psi(x'/2^m)$ shall approach the limit $\psi(0) \equiv 1$. Hence when x' in (9) is replaced by $x'/2^m$, we must select in the right-hand member for sufficiently large values of m that fourth root whose argument is most nearly equal to 0. Then from and after some fixed value $m = m'$ the arguments of the successive fourth roots will be each a fourth of the preceding argument. Hence if we substitute $x'/2^{m'}$ in place of x' for our initial point but still denote it by x', all ambiguity in the determination of $\psi(x'/2^m)$ will be removed. Then by (8) and (10) we have

$$\psi\left(\frac{nx'}{2^m}\right) = [\psi(x')]^{(n/2^m)^2}.$$

Thus our function is uniquely determined over the dense set $\{nx'/2^m\}$ on one half of the line, and a like determination for the other half results from the relation $\psi(-x) = \psi(x)$. Being continuous, the function is thereby completely determined, provided only the values obtained over the dense point-set are consistent with the requirement of continuity.

To establish this consistency choose a value of c such that $e^{cx'^2}$ equals the assigned initial value and has the same argument. By substitution of e^{cx^2} for $\psi(x)$ in (C) it will be found that the equation is satisfied. The function e^{cx^2} also fulfills the requirement that when x is replaced by $x/2$, its argument is divided by 4. We conclude therefore, that unless $\psi(x)$ vanishes along the entire line, it must have the form e^{cx^2}.

Proceed next to the determination of the solution over the complex plane. Should $\psi(x)$ vanish along one line through the origin and be of form e^{cx^2} on another such line, there would be two conflicting values at $x = 0$. Hence if $\psi(x)$ vanishes at a single point, it must vanish over the entire plane. Assume now for $\psi(x)$ arbitrary values different from zero at three initial points x', y', $x' - y'$ which do not lie on a common line through the origin. Its value at $x' + y'$ is then determined by (C). By putting in this equation $y = y'$, $x = x' + ny'$ $(n = 1, 2, \cdots)$ we determine $\psi(x)$ successively at the points $x' + (n+1)y'$, and similarly at $y' + (n+1)x'$. Equation (8) gives $\psi(nx')$ and $\psi(ny')$. Then placing $y = x'$, $x = nx' + my'$ $(n = 1, 2, \cdots)$ in (C) we get $\psi(x)$ at all remaining points of form $nx' + my'$. Its values at $nx' - my'$ may next be found by replacing x, y in (C) by nx', my'. Lastly, $\psi(-nx' \pm my') = \psi(nx' \mp my')$. Thus $\psi(x)$ is uniquely determined over the periodic network $\pm mx' \pm ny'$ in terms of its values at the three initial points.

As before, equation (9) may be used to halve successively the initial arguments x', y' and $x' - y'$, the choice of the fourth root in the right-hand

member being ultimately unique. If now for a sufficiently large value m' we substitute $x'/2^{m'}$, $y'/2^{m'}$, $(x'-y')/2^{m'}$ with the corresponding values of ψ in place of the initial points and values but retain the old notation x', y', $x'-y'$ for the new initial points, all ambiguity disappears. Then $\psi(x)$ will be known uniquely over the dense set of points

$$\frac{nx'}{2^m}+\frac{n'y'}{2^{m'}} \qquad (m, n, m', n' = 0, \pm 1, \pm 2, \cdots),$$

and therefore over the entire plane, provided that the values over this point-set are consistent with the requirement of continuity for $\psi(x)$.

To settle this question we will set up an actual solution of (C) which will take arbitrarily assumed values at the initial points. Put

$$x = u + iv,$$

and take for $\psi(x) \equiv \psi(u+iv)$ the function

$$e^{c_1u^2+c_2uv+c_3v^2}, \tag{11}$$

in which the c_i denote arbitrary constants, real or complex. By direct substitution in (C) it will be found that the functional equation is satisfied. Let e^{k_1}, e^{k_2}, e^{k_3} denote the values assigned arbitrarily at the initial points

$$x' \equiv u_1 + iv_1, \qquad y' \equiv u_2 + iv_2, \qquad \text{and} \qquad x'-y'.$$

The expression (11) will take these values if

$$k_1 = c_1u_1^2 + c_2u_1v_1 + c_3v_1^2,$$

$$k_2 = c_1u_2^2 + c_2u_2v_2 + c_3v_2^2,$$

$$k_3 = c_1(u_1-u_2)^2 + c_2(u_1-u_2)(v_1-v_2) + c_3(v_1-v_2)^2.$$

The determinant of this system of equations for c_1, c_2, c_3 is

$$\begin{vmatrix} u_1^2 & u_1v_1 & v_1^2 \\ u_2^2 & u_2v_2 & v_2^2 \\ (u_1-u_2)^2 & (u_1-u_2)(v_1-v_2) & (v_1-v_2)^2 \end{vmatrix} \equiv (u_1v_2 - v_1u_2)^3,$$

which vanishes only if u_1+iv_1 and u_2+iv_2 lie on a line which passes through the origin. As this is contrary to hypothesis, the expression (11) can be made to take the required values.

Besides our solution $\psi(x)$ there is an equivalent solution, $-\psi(x)$. *Thus the expression* (11) *and its negative gives the general expression for all continuous solutions of* (C) *except the trivial solution* $\psi(x) = 0$. If, in particular, the solution is analytic, it has the form $\psi(x) = \pm e^{cx^2}$.

The functional equation (C) comes a second time into the investigation when a common factor $s(x)$ is sought by which $\phi(x)$, $\psi(x)$ can be modified without ceasing to be a solution of (A). For let them be replaced in either equation of (A) by $s(x)\phi(x)$, $s(x)\psi(x)$ and divide the resulting equation by the same equation before the substitution. Thereby we obtain (C) with $s(x)$ in place of $\psi(x)$. The division becomes impossible for both pairs of equations only if

$$\phi(x+y)=0, \qquad \psi(x-y)=0, \tag{12'}$$

or

$$\phi(x-y)=0, \qquad \psi(x+y)=0. \tag{12''}$$

Hence at most $s(x)$ can fail to satisfy (C) only for exceptional pairs of values (x, y) which satisfy one of these pairs of equations. Now because of the periodicity of their roots neither $\phi(x)$ nor $\psi(x)$ vanishes over any region of the x-plane unless it vanishes everywhere. Hence in the vicinity of any exceptional pair (x, y) there are values for which (C) is satisfied by $s(x)$, whence it follows by continuity that the equation is also satisfied for these exceptional values. We conclude therefore that $\phi(x)$, $\psi(x)$ *can be multiplied by a continuous common factor without ceasing to be a solution of* (A) *when and only when the factor is of form* (11).

3. Solutions with lines of zeros

The case in which $\phi(x)$ possesses one or more lines on which it vanishes identically can be quickly disposed of. Let α denote a zero lying upon the line of zeros through the origin. From (5) it follows that the ratio

$$R^2(x+\alpha) \equiv \phi^2(x+\alpha)/\psi^2(x+\alpha)$$

for fixed x must remain constant when α continuously describes the line of zeros. Consequently $R^2(x)$ remains constant when x describes any straight line parallel to the line of zeros, and this must likewise be true of $R(x)$ which can not change sign abruptly

For convenience suppose the line of zeros not to coincide with the real axis. If necessary, this can be avoided by a rotation of the plane around the origin which in no wise affects the form of (A). Then any solution in which $\phi(x)$, $\psi(x)$ fulfill the above requirement of having a constant ratio on every line parallel to the line of zeros can evidently be constructed as follows. Start with its section $\phi(u)$, $\psi(u)$ along the real axis and construct functions $\bar{\phi}(x)$, $\bar{\psi}(x)$ which coincide with this section along the real axis and are constant on every line parallel to the line of zeros. Clearly $\bar{\phi}(x)$, $\bar{\psi}(x)$ satisfy (A). The solution can then be obtained by modifying these func-

tions by a common factor, which can only be of form (11). The zeros of $\phi(x)$, $\psi(x)$ on the real axis give rise to the lines of zeros.

The following possibilities occur:

(1) $\phi(x)$ possesses only a line of zeros through the origin.

(2) $\phi(x)$ possesses a set of equidistant lines of zeros, while $\psi(x)$ has no zeros.

(3) Both $\phi(x)$ and $\psi(x)$ possess a set of equidistant lines of zeros, the lines for either function being half way between two consecutive lines for the other function.

The real solutions given in § 10 give rise to solutions of the 2d and 3d kinds, while (1) can be obtained from suitable unreal solutions.

Since *every solution with lines of zeros can be constructed in the above very simple manner from a section of some solution along the axis or other line through the origin*, the case is essentially trivial and is solved when the other cases are solved. Hence it need not be considered further.

4. The normalization of the solutions

It may be henceforth supposed that the zeros of $\phi(x)$ form a discrete system, either simply or doubly periodic. Let α be a *primitive* zero, that is, one so situated that no zero lies in the segment $(0, \alpha)$. Place

$$\phi(x+\alpha) = t(x)\phi(x), \tag{13a}$$

where $t(x)$ is a function to be determined. From (5) and (13a) we have

$$\psi(x+\alpha) = \pm t(x)\psi(x). \tag{13b}$$

A comparison of the last two equations shows that the sign in the right-hand member of the latter equation cannot vary with x inasmuch as $\phi(x+\alpha)/\phi(x)$ and $\psi(x+\alpha)/\psi(x)$ are continuous except at the zeros of their respective denominators which do not simultaneously vanish. Then $t(x)$ must be continuous over the entire x-plane.

To determine $t(x)$ replace x by $x+\alpha$ in either equation of (A) and divide either resulting equation by the same equation before the replacement. By application of (13) we obtain in either case

$$t(x+y)t(x-y) = t^2(x). \tag{14}$$

The division fails only for values of the arguments which satisfy (12′) or (12″), and the equation can be extended to these arguments also by a proper limiting passage, as in § 2. Equation (14) gives

$$t(2x)t(0) = t^2(x),$$

and hence on placing $f(x) = t(x)/t(0)$ it can be reduced to the well-known form

$$f(x)f(x') = f(x + x'). \tag{15}$$

Any solution of this equation which is continuous along the real axis and does not vanish identically was shown by Cauchy to have the form e^{cx} along the x-axis, from which it follows that

$$f(u + iv) = f(u)f(iv) = e^{c'u+c''v}.$$

Thus we have as the general solution of (14)

$$t(x) \equiv t(u + iv) = ce^{c'u+c''v},$$

in which c, c', c'' denote arbitrary constants.

Equation (13a) may now be written

$$\phi(u + iv + \alpha) = ce^{c'u+c''v}\,\phi(u + iv) \tag{16a}$$

with a corresponding equation (16b) for $\psi(u + iv)$. The three constants here are not independent but if $\alpha = \alpha' + i\alpha''$, are bound by the relation

$$c = -e^{\frac{1}{2}(c'\alpha'+c''\alpha'')} \tag{17}$$

obtained by putting $u + iv = -\alpha/2$.

Our equations (16) may now be *normalized* by using the property that $\phi(x)$, $\psi(x)$ remain a solution of (A) when modified by an arbitrary factor of form (11). If $\phi(x)$ is replaced by $e^{c_1u^2+c_2uv}\,\phi(x)$, equation (16$a$) thereby becomes

$$e^{c_1(2\alpha'u+\alpha'^2)+c_2(\alpha'v+\alpha''u+\alpha'\alpha'')}\,\phi(u + iv + \alpha' + i\alpha'') = ce^{c'u+c''v}\,\phi(u + iv).$$

Unless* $\alpha' = 0$, the arbitraries c_1, c_2 can be so chosen that the exponential term on the right is canceled out. Then in consequence of (17) our equation reduces to

$$\phi(u + iv + \alpha) = -\phi(u + iv).$$

Thus by normalization (16) may be thrown into one of two forms, either

$$\phi(x + \alpha) = -\phi(x), \qquad \psi(x + \alpha) = -\psi(x), \tag{18}$$

or

$$\phi(x + \alpha) = -\phi(x), \qquad \psi(x + \alpha) = \psi(x). \tag{19}$$

In similar manner, if β is a primitive root of $\psi(x)$, we may place in accord with (6)

* In which case we can obtain the same result by using $e^{c_2uv+c_3v^2}\phi(x)$.

$$\phi(x+\beta) = t(x)\psi(x),$$
$$\psi(x+\beta) = \pm t(x)\phi(x),$$

in which $t(x)$ denotes again a continuous function to be determined. If x is replaced by $x+\beta$ in (A) and either equation after the replacal is divided by the other equation before the replacal, equation (14) is again obtained for $t(x)$. Hence

$$\begin{aligned} \phi(x+\beta) &= be^{b_1u+b_2v}\psi(x), \\ \psi(x+\beta) &= \pm be^{b_1u+b_2v}\phi(x) \end{aligned} \qquad (x = u+iv). \tag{20}$$

Here, as in (13*b*), the sign before b in the second right-hand member is independent of x, and for like reason. From (4) we have $\phi^4(\beta/2) = \psi^4(\beta/2)$, and hence if $\beta = \beta' + i\beta''$, we obtain by putting $x = -\beta/2$

$$b^4 = e^{2(\beta' b_1 + \beta'' b_2)}.$$

By normalizing (20) in the same manner as (16), though not simultaneously unless $\alpha = 2\beta$, we may remove the exponential terms. Thereafter we have

$$\begin{aligned} \phi(x+\beta) &= l\psi(x), \\ \psi(x+\beta) &= \pm l\phi(x) \end{aligned} \qquad (l^4 = 1). \tag{21}$$

These equations are consistent for $x = -\beta/2$ only if $l^2 = \mp 1$. By making a proper choice among the eight equivalent solutions they may be given the definite form:

$$\begin{aligned} \phi(x+\beta) &= \psi(x), \\ \psi(x+\beta) &= -\phi(x). \end{aligned} \tag{22}$$

The last pair of equations as well as the one immediately preceding gives

$$\phi(x+2\beta) = -\phi(x), \qquad \psi(x+2\beta) = -\psi(x).$$

From this it follows that equations (18) and not (19) must be chosen when $\alpha/2$ is a root of $\psi(x)$. Conversely, when (18) holds, $\alpha/2$ must be a root of $\psi(x)$, as is apparent on placing $x = -\alpha/2$ in (18). *Thus the choice between* (18) *and* (19) *is settled by the presence or absence of a root of* $\psi(x)$ *at* $\alpha/2$.

By means of our normalized formulas we may now derive a formula for $\phi(2x)$. Suppose first that $\alpha/2$ is not a root of $\psi(x)$ so that (19) comes into play. Upon replacing x, y, in (1) by $x+\alpha/4$, $x-\alpha/4$ we get

$$\begin{aligned} \phi\left(\frac{\alpha}{2}\right)\phi(2x) = {} & \phi^2\left(x-\frac{3\alpha}{4}\right)\psi^2\left(x+\frac{3\alpha}{4}\right) \\ & - \phi^2\left(x+\frac{3\alpha}{4}\right)\psi^2\left(x-\frac{3\alpha}{4}\right). \end{aligned} \tag{23}$$

Also from (A) we have

$$\pm\phi\left(x\pm\frac{3\alpha}{4}\right)\phi\left(\frac{\alpha}{4}\right)=\phi^2\left(\frac{x}{2}\pm\frac{\alpha}{2}\right)\psi^2\left(\frac{x}{2}\pm\frac{\alpha}{4}\right)-\psi^2\left(\frac{x}{2}\pm\frac{\alpha}{2}\right)\phi^2\left(\frac{x}{2}\pm\frac{\alpha}{4}\right),$$

$$\psi\left(x\pm\frac{3\alpha}{4}\right)\psi\left(\frac{\alpha}{4}\right)=\psi^2\left(\frac{x}{2}\pm\frac{\alpha}{2}\right)\psi^2\left(\frac{x}{2}\pm\frac{\alpha}{4}\right)-\phi^2\left(\frac{x}{2}\pm\frac{\alpha}{2}\right)\phi^2\left(\frac{x}{2}\pm\frac{\alpha}{4}\right).$$

By the use of these we get for one factor of the right-hand member of (23)

$$\begin{aligned}&\phi\left(x+\frac{3\alpha}{4}\right)\psi\left(x-\frac{3\alpha}{4}\right)+\phi\left(x-\frac{3\alpha}{4}\right)\psi\left(x+\frac{3\alpha}{4}\right)\\&=\frac{1}{\phi\left(\frac{\alpha}{4}\right)\psi\left(\frac{\alpha}{4}\right)}\left[\psi^4\left(\frac{x}{2}+\frac{\alpha}{2}\right)-\phi^4\left(\frac{x}{2}+\frac{\alpha}{2}\right)\right]\\&\qquad\cdot\left[\phi^2\left(\frac{x}{2}-\frac{\alpha}{4}\right)\psi^2\left(\frac{x}{2}+\frac{\alpha}{4}\right)-\phi^2\left(\frac{x}{2}+\frac{\alpha}{4}\right)\psi^2\left(\frac{x}{2}-\frac{\alpha}{4}\right)\right]\\&=\frac{-\psi(x)\phi(x)\phi\left(\frac{\alpha}{2}\right)}{\phi\left(\frac{\alpha}{4}\right)\psi\left(\frac{\alpha}{4}\right)}.\end{aligned}\tag{24}$$

The other factor of the right-hand member by virtue of (19) is identical with the first factor, after x is replaced by $x+\alpha/2$ in the latter factor. Hence we have

$$\phi(2x)=\frac{\phi\left(\frac{\alpha}{2}\right)}{\phi^2\left(\frac{\alpha}{4}\right)\psi^2\left(\frac{\alpha}{4}\right)}\phi(x)\psi(x)\phi\left(x+\frac{\alpha}{2}\right)\psi\left(x+\frac{\alpha}{2}\right).$$

But for $x=\alpha/2$ equation (24) gives

$$2\phi\left(\frac{\alpha}{4}\right)\psi\left(\frac{\alpha}{4}\right)=\frac{\phi^2\left(\frac{\alpha}{2}\right)\psi\left(\frac{\alpha}{2}\right)}{\phi\left(\frac{\alpha}{4}\right)\psi\left(\frac{\alpha}{4}\right)},$$

so that the preceding equation takes the final form

$$\phi\left(\frac{\alpha}{2}\right)\psi\left(\frac{\alpha}{2}\right)\phi(2x)=2\phi(x)\psi(x)\phi\left(x+\frac{\alpha}{2}\right)\psi\left(x+\frac{\alpha}{2}\right).\tag{25}$$

The reader will observe the identity in structure with the familiar formula for the Jacobian function $\vartheta_1(2x)$. It holds when $\alpha/2$ is not a root of $\psi(x)$ and the normalization of the functions is made with reference to the root α.

Suppose, on the other hand, that $\beta \equiv \alpha/2$ is a root of $\psi(x)$, and select from the equivalent normalized solutions one for which (22) is valid. Then by (22) we have

$$\phi\left(\frac{\alpha}{2}\right) = \psi(0) = 1, \qquad \phi\left(\frac{\alpha}{4}\right) = \psi\left(\frac{\alpha}{4}\right), \tag{26}$$

and by (22) and (4)

$$\phi(2x) = -\psi\left(2x + \frac{\alpha}{2}\right) = -\psi^4\left(x + \frac{\alpha}{4}\right) + \phi^4\left(x + \frac{\alpha}{4}\right).$$

By means of (26) and the relation

$$\phi(x)\psi(x) = \phi(x)\phi\left(x + \frac{\alpha}{2}\right) = \phi^2\left(x + \frac{\alpha}{4}\right)\psi^2\left(\frac{\alpha}{4}\right) - \phi^2\left(\frac{\alpha}{4}\right)\psi^2\left(x + \frac{\alpha}{4}\right)$$

the last equation may be thrown into the form

$$\phi(2x) = \frac{\phi(x)\psi(x)}{\phi^2\left(\frac{\alpha}{4}\right)}\left[\phi^2\left(x + \frac{\alpha}{4}\right) + \psi^2\left(x + \frac{\alpha}{4}\right)\right]. \tag{27}$$

When $\phi^2 + \psi^2 \equiv 1$, this formula is identical in structure with that for $\sin 2x$.

5. On the existence of zeros of $\psi(x)$

The existence of a zero of $\psi(x)$, and hence of a simply or doubly periodic system of zeros like that of $\phi(x)$, can now be established.

Suppose first the zeros of $\phi(x)$ form a doubly periodic network, and let α, α' be a primitive pair of zeros. If $\alpha/2$ is not a root of $\psi(x)$, we may make use of the normalized equation (25). For $x = \alpha'/2$ this becomes

$$0 = \phi\left(\frac{\alpha'}{2}\right)\psi\left(\frac{\alpha'}{2}\right)\phi\left(\frac{\alpha + \alpha'}{2}\right)\psi\left(\frac{\alpha + \alpha'}{2}\right).$$

Hence $\psi(x)$ must vanish at $\alpha'/2$ or $(\alpha + \alpha')/2$ if it does not vanish at $\alpha/2$.

When the zeros of $\phi(x)$ form only a simply periodic system, the existence of a zero of $\psi(x)$ can be readily established in the following manner similar to that in which a zero was proved to exist for either $\phi(x)$ or $\psi(x)$ in § 1. Suppose, if possible, $\psi(x)$ to have no root. Then by (19) we have

$$R(x + \alpha) = -R(x). \tag{28}$$

Through $\alpha/2$ and $-\alpha/2$ in the x-plane draw two parallel lines, and denote by S a half of the resulting strip which lies to one side of the segment joining $\pm\alpha/2$. Consider the representation of S upon the R-plane effected through $R(x)$. As seen in § 1, it is impossible for $R(x)$ to assume any of the

values $\pm 1, \pm i$ if $\psi(x)$ has no zero, and (B) has then always a sense. If $R(x)$ should take the same value at two different points x, y of S, it would follow from (B) that either $x+y$ or $x-y$ would be a root of $R(x)$ and hence would be equal to 0 or a multiple of α, which necessarily is $\pm\alpha$ when x, y lie in S. But clearly we cannot have $x-y=\pm\alpha$ because of (28), nor $x+y=0$ because $R(-x)=-R(x)$. Hence $R(x)$ will build the half-strip by a continuous 1–1 correspondence upon a simply connected region of the R-plane whose boundary passes through the origin. The other half-strip is built upon a like region which is the symmetrical image of the first with respect to the origin since $R(-x)=-R(x)$. A circle of radius $\alpha/2$ about the origin of the x-plane by the same reasoning as was applied to S will be built by continuous 1–1 correspondence upon a region inclosing the origin of the R-plane. Take in the latter region a circle C' of radius ϵ about the origin whose image will be a very small vicinity C of $x=0$. Then $|R(x)|$ by (28) can be equal to or less than ϵ only when x falls in C or a congruent vicinity of one of the points $x=\pm n\alpha$.

Denote the R-image of S by S'. Any segment parallel to the base of S and terminated by its parallel sides will be built upon some sort of a line of the R-plane whose terminal points lie without C' and which by (28) are symmetrically situated with respect to the origin. As the segment recedes to infinity, either its image will sweep over a region which extends to infinity or there will be frontier points L of the region other than $\pm 1, \pm i$. In the vicinity of any such point L we can select within S' a set of R-points with the limit L, for which the corresponding x-points recede to infinity. From these we can take two points $R(x), R(y)$ arbitrarily close to L in such wise that $|y|$ will be as large as we please relatively to $|x|$, and in consequence both $|x+y|$ and $|x-y|$ will be extremely large. If, furthermore, $R(x)$ and $R(y)$ are sufficiently close to L and therefore to each other, it follows by (B) that either $|R(x+y)|$ or $|R(x-y)|$ may be made arbitrarily small. Accordingly, either $x+y$ or $x-y$ may be made to fall within C, or within a congruent vicinity of one of the points $\pm\alpha$. But obviously this is impossible if both $|x+y|$ and $|x-y|$ are very large. From this contradiction it results that S' must stretch out to infinity. Then it must contain points $R(x)$ such that

$$1/|R(x)|<\epsilon,$$

and hence points $R(x), R(y)$ for which $R(x)R(y)=1$. It follows then by (B) that either $R(x+y)$ or $R(x-y)$ must be infinite. But this is impossible unless $\psi(x+y)=0$ or $\psi(x-y)=0$. Consequently the supposition that $\psi(x)$ has no zero is untenable. Having one zero, it has a doubly or simply periodic set of zeros simultaneously with $\phi(x)$.

6. The allied solutions

We seek next to find non-equivalent solutions whose first components $\phi(x)$ differ only by a constant factor.

Suppose $\phi(x)$, $\psi(x)$ to be any solution normalized according to (18) or (19). Then it may be shown quickly that

$$(29) \qquad \phi_2(x) \equiv \frac{\phi(x)}{\phi\left(\frac{\alpha}{2}\right)}, \qquad \psi_2(x) \equiv \frac{\phi\left(x+\frac{\alpha}{2}\right)}{\phi\left(\frac{\alpha}{2}\right)} \equiv \phi_2\left(x+\frac{\alpha}{2}\right)$$

is also a solution. For let these functions be substituted in (A) in place of $\phi(x)$, $\psi(x)$. The equations thereby become

$$(30) \quad \phi^2\left(\frac{\alpha}{2}\right)\phi(x+y)\phi(x-y) = \phi^2(x)\phi^2\left(y+\frac{\alpha}{2}\right) - \phi^2(y)\phi^2\left(x+\frac{\alpha}{2}\right),$$

$$(31) \quad \phi^2\left(\frac{\alpha}{2}\right)\phi\left(x+y+\frac{\alpha}{2}\right)\phi\left(x-y+\frac{\alpha}{2}\right) = \phi^2\left(x+\frac{\alpha}{2}\right)\phi^2\left(y+\frac{\alpha}{2}\right) - \phi^2(x)\phi^2(y).$$

The first of these passes into the second when x is replaced by $x + \alpha/2$. It suffices then to verify the truth of the former. Now

$$(32) \quad \phi^2\left(x+\frac{\alpha}{2}\right) = -\phi\left(x+\frac{\alpha}{2}\right)\phi\left(x-\frac{\alpha}{2}\right) = -\phi^2(x)\psi^2\left(\frac{\alpha}{2}\right) + \phi^2\left(\frac{\alpha}{2}\right)\psi^2(x).$$

The introduction of this expression and of the like expression for $\phi^2(y+\alpha/2)$ into (30) converts it into (1). Accordingly (29) furnishes a solution of (A).

It may next be shown that if $\alpha/2$ is not a zero of $\psi(x)$, then

$$(33) \qquad \frac{\phi(x)}{\sqrt{i}\psi\left(\frac{\alpha}{2}\right)}, \qquad \frac{\psi\left(x+\frac{\alpha}{2}\right)}{\psi\left(\frac{\alpha}{2}\right)}$$

also constitute a solution of (A). For by virtue of (19) we have

$$(34) \quad \psi^2\left(x+\frac{\alpha}{2}\right) = \psi\left(x+\frac{\alpha}{2}\right)\psi\left(x-\frac{\alpha}{2}\right) = \psi^2(x)\psi^2\left(\frac{\alpha}{2}\right) - \phi^2(x)\phi^2\left(\frac{\alpha}{2}\right).$$

If now the functions (33) are substituted in (A) in place of $\phi(x)$, $\psi(x)$, two

equations are obtained, of which the first becomes identical with (1) by the aid of (34). The second is

$$\psi^2\left(\frac{\alpha}{2}\right)\psi\left(x+y+\frac{\alpha}{2}\right)\psi\left(x-y+\frac{\alpha}{2}\right) = \psi^2\left(x+\frac{\alpha}{2}\right)\psi^2\left(y+\frac{\alpha}{2}\right)+\phi^2(x)\phi^2(y).$$

If x is here replaced by $x+\alpha/2$, the equation reduces at once to (2) through the use of (32), (34), and (19). Hence (33) gives a solution, as stated.

Seek next every possible solution

$$\phi_i(x) \equiv c_i\,\phi(x), \qquad \psi_i(x), \tag{35}$$

having the same first component save for a constant factor c_i. It should be carefully noted that *this entire group of solutions is simultaneously normalized,* inasmuch as a normalization of the ϕ-component carries with it a normalization of the corresponding ψ. But equation (18) may hold for one solution and (19) for another, and a corresponding remark applies to (21).

By substitution of (35) in (1) and then by combination of the resulting equation with (1) we get

$$\frac{\phi^2(x)}{\psi^2(x)-\psi_i^2(x)} = \frac{\phi^2(y)}{\psi^2(y)-\psi_i^2(y)}.$$

As the arguments of the two members are independent, each member is a constant. Hence

$$\psi_i^2(x) = \psi^2(x) + k_i^2\,\phi^2(x),$$

in which k_i^2 is a constant to be subsequently determined. Take now (29) as the *primary* solution, for which we have

$$\phi_2\left(\frac{\alpha}{2}\right)=1, \qquad \psi_2\left(\frac{\alpha}{2}\right)=0, \qquad \phi_2\left(\frac{\alpha}{4}\right)=\psi_2\left(\frac{\alpha}{4}\right). \tag{36}$$

Our preceding equation becomes

$$\psi_i^2(x) = \psi_2^2(x) + k_i^2\,\phi_2^2(x) \equiv \phi_2^2\left(x+\frac{\alpha}{2}\right) + k_i^2\,\phi_2^2(x). \tag{37}$$

For $x=\alpha/2$ this gives

$$k_i^2 = \psi_i^2\left(\frac{\alpha}{2}\right). \tag{38}$$

If $\psi_i(x)$ vanishes at $\alpha/2$, we have $k_i=0$, $c_i^4=1$ and obtain the primary solution or an equivalent solution. Henceforth it may be assumed that $\psi_i(x)$ does not vanish at $\alpha/2$ so that we have $k_i \neq 0$ and also, by (19), $\psi_i(\alpha)=1$. Then on replacing ψ by ψ_i in (4) and putting $x=\alpha/2$ we have

$$1 = k_i^4 - c_i^4. \tag{39}$$

If the functions (35) with $\phi(x)$ taken equal to $\phi_2(x)$ are substituted in the two equations of (A), the first is satisfied regardless of the values of k_i and c_i in consequence of (37). If the second is squared and modified with the aid of (37) and (39), it becomes

$$\begin{aligned}\left[\phi_2^2\left(x+y+\frac{\alpha}{2}\right)+k_i^2\,\phi_2^2(x+y)\right]\cdot\left[\phi_2^2\left(x-y+\frac{\alpha}{2}\right)+k_i^2\,\phi_2^2(x-y)\right]\\ =\left[\phi_2^2\left(x+\frac{\alpha}{2}\right)\phi_2^2\left(y+\frac{\alpha}{2}\right)+\phi_2^2(x)\,\phi_2^2(y)\right.\\ \left.+k_i^2\left\{\phi_2^2(x)\,\phi_2^2\left(y+\frac{\alpha}{2}\right)+\phi_2^2(y)\,\phi_2^2\left(x+\frac{\alpha}{2}\right)\right\}\right]^2.\end{aligned} \tag{40}$$

For $x = y = \alpha/4$ this furnishes for k_i^2 the quadratic equation

$$k_i^4+\left(2-\frac{1}{4\phi_2^8\left(\frac{\alpha}{4}\right)}\right)k_i^2+1=0. \tag{41}$$

By a somewhat laborious calculation it will next be shown that either of the two values of k_i^2 will reduce (40) to an identity. Let k_i^4 be first eliminated by (41). Then the terms independent of k_i^2 will cancel by (31), (30), and (36). Equating the coefficients of k_i^2 we have left as the identity to be verified,

$$\begin{aligned}\phi_2^2(x+y)\,\phi_2^2\left(x-y+\frac{\alpha}{2}\right)+\phi_2^2(x-y)\,\phi_2^2\left(x+y+\frac{\alpha}{2}\right)\\ =2\left[\phi_2^2\left(x+\frac{\alpha}{2}\right)\phi_2^2\left(y+\frac{\alpha}{2}\right)+\phi_2^2(x)\,\phi_2^2(y)\right]\\ \cdot\left[\phi_2^2(x)\,\phi_2^2\left(y+\frac{\alpha}{2}\right)+\phi_2^2(y)\,\phi_2^2\left(x+\frac{\alpha}{2}\right)\right]\\ -\left(2-\frac{1}{4\phi_2^8\left(\frac{\alpha}{4}\right)}\right)\left\{\left[\phi_2^2(x)\,\phi_2^2\left(y+\frac{\alpha}{2}\right)+\phi_2^2(y)\,\phi_2^2\left(x+\frac{\alpha}{2}\right)\right]^2\right.\\ \left.-\phi_2^2(x+y)\,\phi_2^2(x-y)\right\}.\end{aligned} \tag{42}$$

By (30) the last term reduces to

$$-4\left[2-\frac{1}{4\phi_2^8\left(\frac{\alpha}{4}\right)}\right]\phi_2^2(x)\,\phi_2^2(y)\,\phi_2^2\left(x+\frac{\alpha}{2}\right)\phi_2^2\left(y+\frac{\alpha}{2}\right).$$

To verify the identity we will eliminate the arguments of the left-hand member, introducing ultimately in their place the same arguments as appear on the right. To this end replace x, y in (30) and (31) by $x + \alpha/4$, $y - \alpha/4$. Then squaring, adding, and using the relation

$$\phi_2^2\left(x - \frac{\alpha}{4}\right) = \psi_2^2\left(x + \frac{\alpha}{4}\right),$$

we obtain the left-hand member of (42) in the form

$$\left[\phi_2^4\left(x + \frac{\alpha}{4}\right) + \psi_2^4\left(x + \frac{\alpha}{4}\right)\right] \cdot \left[\phi_2^4\left(y + \frac{\alpha}{4}\right) + \psi_2^4\left(y + \frac{\alpha}{4}\right)\right] - 4\phi_2^2\left(x + \frac{\alpha}{4}\right)\psi_2^2\left(x + \frac{\alpha}{4}\right)\phi_2^2\left(y + \frac{\alpha}{4}\right)\psi_2^2\left(y + \frac{\alpha}{4}\right). \tag{43}$$

Now with the help of the third of relations (36) we have

$$\begin{aligned}\phi_2^4\left(\frac{\alpha}{4}\right)\left[\phi_2^4\left(x + \frac{\alpha}{4}\right) + \psi_2^4\left(x + \frac{\alpha}{4}\right)\right] &= \left[\phi_2^2\left(x + \frac{\alpha}{4}\right)\psi_2^2\left(\frac{\alpha}{4}\right) - \psi_2^2\left(x + \frac{\alpha}{4}\right)\phi_2^2\left(\frac{\alpha}{4}\right)\right]^2 \\ &\quad + 2\phi_2^4\left(\frac{\alpha}{4}\right)\phi_2^2\left(x + \frac{\alpha}{4}\right)\psi_2^2\left(x + \frac{\alpha}{4}\right) \\ &= \phi_2^2\left(x + \frac{\alpha}{2}\right)\phi_2^2(x) + 2\phi_2^4\left(\frac{\alpha}{4}\right)\phi_2^2\left(x + \frac{\alpha}{4}\right)\psi_2^2\left(x + \frac{\alpha}{4}\right)\end{aligned}$$

and also

$$\begin{aligned}\phi_2\left(x + \frac{\alpha}{4}\right)\psi_2\left(x + \frac{\alpha}{4}\right) &= -\phi_2\left(x + \frac{\alpha}{4}\right)\phi_2\left(x - \frac{\alpha}{4}\right) \\ &= \phi_2^2\left(\frac{\alpha}{4}\right)[\psi_2^2(x) - \phi_2^2(x)] = \phi_2^2\left(\frac{\alpha}{4}\right)\left[\phi_2^2\left(x + \frac{\alpha}{2}\right) - \phi_2^2(x)\right].\end{aligned}$$

By means of these two equations the argument $x + \alpha/4$ and similarly $y + \alpha/4$ may be eliminated from (43), whereby it becomes identical with the right-hand member of (42). Thus (40) becomes an identity when k_i^2 is either root of (41), and consequently the second of equation (A) is satisfied by (35) *after* it has been squared.

It remains to show that the last mentioned equation is also satisfied before being squared. Now this is obvious for $x = y = 0$, and when x, y are continuously varied, its two members must continue to remain equal so long as x, y do not pass through values which make the left-hand member vanish; i. e., exceptional values for which either

$$\psi_i(x+y)=0 \quad \text{or} \quad \psi_i(x-y)=0.$$

For fixed x or for fixed y the values of y and x respectively which satisfy either equation form a set of isolated values.* Let x', y' be any pair of values for which neither $\psi_i(x'+y')=0$ nor $\psi_i(x'-y')=0$. Then by continuous variation it is possible to pass from $x=0$, $y=0$ to $x=x'$, $y=y'$ without passing through exceptional values of x, y. For first, x may be held equal to 0 (or a very small ϵ) while y is varied from 0 to y' in such manner as to avoid exceptional values, and then while y remains at y', x may be varied in like manner from 0 (or ϵ) to x'. The second of equations (A) then holds for $x=x'$, $y=y'$ before being squared and hence it is always valid. We conclude therefore that for the two values of k_i^2 furnished by (41) the functions (35) will satisfy (A).

It has thus been shown that *with* $\phi_2(x)$, $\psi_2(x)$ *we may ally two and only two solutions of form* $\phi_i(x)=c_i\phi_2(x)$, $\psi_i(x)$, to which the indices $i=1$, 3 will be assigned. Four allied functions are thereby connected, as in the theory of the Jacobi ϑ-functions. From (38) and (41) we get

$$1=k_1^2k_3^2=\psi_1^2\left(\frac{\alpha}{2}\right)\psi_3^2\left(\frac{\alpha}{2}\right). \tag{44}$$

Then from (37) and (44) it follows that

$$\psi_1^2(x)\equiv\frac{\psi_3^2\left(x+\frac{\alpha}{2}\right)}{\psi_3^2\left(\frac{\alpha}{2}\right)}, \qquad \psi_3^2(x)\equiv\frac{\psi_1^2\left(x+\frac{\alpha}{2}\right)}{\psi_1^2\left(\frac{\alpha}{2}\right)}. \tag{45}$$

Furthermore, since every $\psi(x)$ possesses zeros whose doubles are zeros of $\phi(x)$ and since $\psi_1(x)$, $\psi_3(x)$ do not vanish at $\alpha/2$, it follows that $\phi_2(x)$ must have two independent sets of zeros, $\pm n\alpha$ and $\pm n\alpha'$. Then simultaneously the three allied $\psi_i(x)$ have each a doubly periodic set of zeros, $\psi_2(x)$ vanishing at $\alpha/2$, a second at $\alpha'/2$, and the remaining one by (45) at $(\alpha+\alpha')/2$.

7. The simply periodic case

While in general there are three allied solutions, an exception occurs when the two roots of the quadratic (41) are equal, namely when

$$\phi_2\left(\frac{\alpha}{4}\right)=\pm\frac{1}{\sqrt{2}}, \qquad \pm\frac{i}{\sqrt{2}}, \qquad \pm\frac{i+1}{2}. \tag{46}$$

* As $\phi_i(x)$, $\psi_i(x)$ satisfy equations (2) when squared, the argument of § 1 applies without change to show that the double of a root β of $\psi_i(x)$ is a root of $\phi_i(x)$. As the roots of $\phi_i(x)\equiv c_i\phi_2(x)$ are isolated, the same must be true of the roots of $\psi_i(x)$.

We have then $k_i^2 = 1$, $c_i = 0$, and $\phi_i(x) \equiv 0$ $(i = 1, 3)$. Thus two allied solutions degenerate, and by § 2 the function

$$\psi_i^2(x) = \phi_2^2(x) + \phi_2^2\left(x + \frac{\alpha}{2}\right) = \phi_2^2(x) + \psi_2^2(x) \qquad (i = 1, 3)$$

becomes merely an expression of form (11). By the removal of this expression as a common factor from $\phi_2^2(x)$, $\psi_2^2(x)$, our last equation reduces to

$$1 = \phi^2(x) + \psi^2(x), \tag{47}$$

where the subscripts are now discarded as useless. The preceding equation for $\psi_i^2(x)$ shows that the factor removed was one which remains unaltered when x is replaced by $x + \alpha/2$. Its removal therefore in no wise affects the normalization (22) for $\phi_2(x)$ and $\psi_2(x)$ with respect to $\beta = \alpha/2$, which accordingly survives for $\phi(x)$, $\psi(x)$ in (47).

By means of (47) the form of our functional equations (A) may be simplified. By elimination of ψ^2 they become

$$\phi(x + y)\phi(x - y) = \phi^2(x) - \phi^2(y),$$
$$\psi(x + y)\psi(x - y) = 1 - \phi^2(x) - \phi^2(y).$$

The last equation gives

$$\psi(2x) = 1 - 2\phi^2(x).$$

Then by combination of the preceding equations and the substitution of x, y for $x + y, x - y$ we have

$$\psi(x)\psi(y) - \phi(x)\phi(y) = \psi(x + y), \tag{D1}$$

whence it follows by the substitution of $x + \alpha/2$ for x and by use of (22) that

$$\phi(x)\psi(y) + \phi(y)\psi(x) = \phi(x + y). \tag{D2}$$

Thus for the special case before us (A) may be replaced by the familiar equations (D), but with a restriction to be immediately specified.

To obtain the general continuous solution of (D) place

$$f(x) = \psi(x) \pm i\phi(x).$$

Then (D) gives the equation

$$f(x + y) = f(x)f(y),$$

of which the general solution ($f(x) \equiv 0$ excepted) is

$$f(u + iv) \equiv f(u)f(iv) = e^{d_1 u + d_2 v}.$$

Hence

$$\psi(x) + i\phi(x) = e^{d_1 u + d_2 v}$$
$$\psi(x) - i\phi(x) = e^{d_3 u + d_4 v} \qquad (x = u + iv)$$

and

$$\phi(x) = \frac{1}{2i}(e^{d_1u+d_2v} - e^{d_3u+d_4v}),$$

$$\psi(x) = \frac{1}{2}(e^{d_1u+d_2v} + e^{d_3u+d_4v}) \qquad (x = u + iv). \tag{48}$$

By substitution it will be found that (D) is satisfied by (48) regardless of the values of the constants, and accordingly (48) furnishes the general solution of (D).

The condition (47) requires in addition that we make $d_3 = -d_1$, $d_4 = -d_2$, which indeed expresses the necessary and sufficient condition that the functions (48) shall be the one odd and the other even. Then the familiar subtraction as well as addition formulas are fulfilled, and hence equations (A). *Thus in the simply periodic case the general solution of* (A), *save for an arbitrary factor of form* (11), *is expressible in the form*

$$\phi(x) = \frac{1}{2i}(e^{d_1u+d_2v} - e^{-(d_1u+d_2v)}),$$

$$\psi(x) = \frac{1}{2}(e^{d_1u+d_2v} + e^{-(d_1u+d_2v)}), \tag{49}$$

or as an "equivalent" solution.

The zeros of $\phi(x)$, $\psi(x)$ lie at the points for which $e^{2d_1u+2d_2v}$ is equal to $+1$ or -1 respectively. Put $d_1 = d_1' + id_1''$, $d_2 = d_2' + id_2''$. Then unless $\Delta \equiv d_1' d_2'' - d_2' d_1'' = 0$, there will be a line through the origin on which $\phi(x)$, $\psi(x)$ possess each a simply periodic set of zeros which alternate with one another. An exception occurs for any solution having $\Delta = 0$. Then there is a line through the origin along which $\phi(x)$ everywhere vanishes, while $\psi(x)$ does not vanish at all unless $d_1' = d_2' = 0$. In this latter case a set of equidistant parallel lines exist along which $\phi(x)$ everywhere vanishes (cf. § 3) while $\psi(x)$ vanishes along the lines half way between consecutive vanishing lines for $\phi(x)$.

In conclusion, it should be pointed out that not all the values given by (46) can be taken by $\phi_2(x)$. In fact, if we put $x = \alpha/4$ in (47) we have

$$\phi^2(\alpha/4) = 1/2$$

since $\psi^2(\alpha/4) = \phi^2(3\alpha/4) = \phi^2(\alpha/4)$. Now the factor which was removed in passing from $\phi_2^2(x)$ to $\phi^2(x)$ was one of form (11) which had the property of remaining unaltered when x was replaced by $x + \alpha/2$ and hence, if

$$\alpha = \alpha' + i\alpha'',$$

it may be written under the form

$$e^{c_1(a'^2u^2-2a'a''uv+a''^2v^2)}.$$

For $u + iv = \alpha/4$ this takes the value 1. Hence only the first pair of values given in (46) is admissible for $\phi_2(x)$. The second set $\pm i/\sqrt{2}$ gives the value of $\phi(\alpha/4)$ for certain equivalent solutions, while the third set is altogether to be excluded. We conclude therefore that *the necessary and sufficient condition that a solution* $\phi(x)$, $\psi(x)$ *shall have a simply periodic set of zeros is that* the zeros of $\phi(x)$ shall be discrete (i. e., not form continuous lines) and *that the normalized function* $\phi_2(x) \equiv \phi(x)/\phi(\alpha/2)$ *shall have the value* $\pm 1/\sqrt{2}$ *for* $x = \alpha/4$, *when* α *is the primitive zero of* $\phi(x)$ *with respect to which it is normalized.* The conditions are indeed necessary, for otherwise, as we have seen, there will be a system of three allied solutions, each with a doubly periodic system of zeros.

8. The doubly periodic case

In § 4 the normalization of the solution was made with reference to any primitive root α of $\phi(x)$. In the ordinary theory of trigonometric or ϑ-functions this root has the position π. To this position α may be brought by an appropriate affine transformation of the x-plane with real coefficients:

$$\begin{aligned} u' &= au + bv, \\ v' &= cu + dv. \end{aligned} \qquad \begin{pmatrix} x = u + iv, \\ x' = u' + iv' \end{pmatrix}. \tag{50}$$

Any such transformation converts any four arguments x, y, $x - y$, $x + y$ into a like quadruple x', y', $x' - y'$, $x' + y'$, and hence the functions $\phi(x)$, $\psi(x)$ after transformation satisfy (A). By such transformation there is connected with any one solution a family of essentially equivalent solutions which are merely the continuous distortion of that solution through affine transformations of the x-plane which leave the origin unaltered. In particular, *with any analytic solution there is thus connected a family of non-analytic solutions.* There exist, however, non-analytic solutions of (A) which are not connected in this manner with analytic solutions, as for example those of § 3. In the simply periodic case of § 7 it is easy to verify that when $\Delta \neq 0$, the solution is reducible by a real affine transformation (50) to the form

$$\phi(x) = \sin \mu x, \qquad \psi(x) = \cos \mu x,$$

in which μ is a constant, real or complex. Except for a factor (11) all solutions of § 7 which do not fall under § 3 are therefore either analytic or the affine distortions of analytic solutions. It seems probable that the like is true also for the doubly periodic case.

In considering henceforth the doubly periodic case we will suppose a primi-

tive pair of zeros brought by an affine transformation to positions $\alpha = \pi$, $2\beta = i \log q$ on the real and imaginary axis respectively. In effecting the normalization of $\phi(x)$ with respect to α in § 4 only a part of the arbitrary factor (11) was employed and we still have the liberty to modify $\phi(x)$, $\psi(x)$ by a factor $e^{c_3 v^2}$, which in no wise disturbs that normalization if α is real. Consider a solution $\phi(x)$, $\psi(x)$ for which β is a root of $\psi(x)$. By the introduction of said factor the first equation of (20) becomes

$$e^{c_3(2\beta'' v + \beta''^2)}\,\phi(x+\beta) = be^{b_1 u + b_2 v}\,\psi(x) \qquad (\beta = \beta' + i\beta''),$$

and hence by proper choice of c_3 we may bring (20) to the form

$$\begin{aligned} \phi(x+\beta) &= b'\,e^{b_1 x}\,\psi(x), \\ \psi(x+\beta) &= \pm\, b'\,e^{b_1 x}\,\phi(x). \end{aligned} \tag{51}$$

Here b', b_1 must be conditioned by the requirement that the two equations shall be consistent for $x = -\beta/2$, which gives

$$\pm\, b'^2\,e^{-b_1 \log q/2} = -1.$$

Replacing x in the first equation of (51) by $x + \alpha \equiv x + \pi$ and then comparing it with the same equation before replacement, we obtain with the aid of (18) or (19) either $e^{b_1 \pi} = +1$ or $e^{b_1 \pi} = -1$, and therefore $b_1 = ni$, where n is some integer, positive, negative, or zero. Equations (51) now take the form:

$$\begin{aligned} \phi\left(x + \frac{i \log q}{2}\right) &= \pm \frac{e^{inx}}{q^{n/4}}\psi(x), \\ \psi\left(x + \frac{i \log q}{2}\right) &= \mp \frac{e^{inx}}{q^{n/4}}\phi(x), \end{aligned} \tag{52}$$

or

$$\begin{aligned} \phi\left(x + \frac{i \log q}{2}\right) &= \pm \frac{ie^{inx}}{q^{n/4}}\psi(x), \\ \psi\left(x + \frac{i \log q}{2}\right) &= \pm \frac{ie^{inx}}{q^{n/4}}\phi(x). \end{aligned} \tag{53}$$

The four alternatives offered by these two sets of formulas are realized simultaneously by equivalent solutions of (A).

When the functions are analytic, we cannot have $n = 0$, for then they would have 2π and $4i \log q$ as periods, which is impossible for analytic functions everywhere finite. If $n \neq 0$, the form of our formulas may be simplified to accord with that found in the theory of the ϑ-functions. Replace x by x/n, and $\phi(x/n)$, $\psi(x/n)$, q^n by $\phi(x)$, $\psi(x)$, q. Formulas (52) and (53) result with n now equal to 1.

It has been already noted that the allied solutions are simultaneously normalized. In particular, equations (18) hold for $\phi_2(x)$, $\psi_2(x)$ and (19) for $\phi_i(x)$, $\psi_i(x)$, $(i = 1, 3)$.

The theory of the ϑ-functions furnishes the following allied solutions of (A),

$$\text{(II)} \quad \phi_2(x) = \frac{\vartheta_1(x)}{\vartheta_1\left(\frac{\pi}{2}\right) = \vartheta_2(0)} \equiv \sqrt[4]{(e_1 - e_3)(e_1 - e_2)}\, e^{-2\eta_1\omega_1 x^2}\sigma(u),$$

$$\psi_2(x) = \frac{\vartheta_2(x)}{\vartheta_2(0)} \equiv e^{-2\eta_1\omega_1 x^2}\sigma_1(u),$$

$$\text{(I)} \quad \phi_1(x) = \frac{\vartheta_1(x)}{\vartheta(0)} \equiv \sqrt[4]{(e_3 - e_1)(e_3 - e_2)}\, e^{-2\eta_1\omega_1 x^2}\sigma(u),$$

$$\psi_1(x) = \frac{\vartheta(x)}{\vartheta(0)} \equiv e^{-2\eta_1\omega_1 x^2}\sigma_3(u),$$

$$\text{(III)} \quad \phi_3(x) = \frac{\vartheta_1(x)}{\sqrt{i}\vartheta_3(0)} \equiv \sqrt[4]{(e_2 - e_1)(e_2 - e_3)}\, e^{-2\eta_1\omega_1 x^2}\sigma(u),$$

$$\psi_3(x) = \frac{\vartheta_3(x)}{\vartheta_3(0) \equiv \vartheta\left(\frac{\pi}{2}\right)} \equiv e^{-2\eta_1\omega_1 x^2}\sigma_2(u),$$

where $u = 2\omega_1 x$. The substitution of these pairs of functions in (A) gives, in fact, six fundamental relations between these functions. Instead of regarding these functions as an application of our theory, we may conceive I–III as defining four ϑ-functions in terms of our allied solutions. For the constants of our preceding theory we have the equivalents

$$c_1 = \frac{\vartheta_2(0)}{\vartheta(0)}, \qquad c_3 = \frac{\vartheta_2(0)}{\sqrt{i}\vartheta_3(0)}, \qquad k_1^2 = \psi_1^2\left(\frac{\pi}{2}\right) = \frac{\vartheta^2\left(\frac{\pi}{2}\right)}{\vartheta^2(0)} = \frac{\vartheta_3^2(0)}{\vartheta^2(0)},$$

$$k_3^2 = \psi_3^2\left(\frac{\pi}{2}\right) = \frac{\vartheta_3^2\left(\frac{\pi}{2}\right)}{\vartheta_3^2(0)}.$$

Then by (44) we get

$$\vartheta_3^2\left(\frac{\pi}{2}\right) = \vartheta^2(0).$$

The application of (39) gives then the fundamental relation:

$$\vartheta_3^4(0) = \vartheta^4(0) + \vartheta_2^4(0).$$

To make the relations between the ϑ-functions thus defined identical with the ordinary ϑ-relations it will be necessary to make a proper selection from

the three allied sets of equivalent solutions. The choice of $\phi_2(x)$, $\psi_2(x)$ has already been made definite by the requirement that they shall be equal to 1 at $x = \pi/2$ and $x = 0$ respectively. For $\phi_1(x)$, $\psi_1(x)$ select the allied solution which has a zero at $i \log q/2$ and which satisfies equations of form (53) with the lower or negative signs in the right-hand members. As the remaining pair of allied functions take in accord with (33)

$$\phi_3(x) = \frac{\phi_1(x)}{\sqrt{i}\,\psi_1\left(\frac{\pi}{2}\right)}, \qquad \psi_3(x) = \frac{\psi_1\left(x + \frac{\pi}{2}\right)}{\psi_1\left(\frac{\pi}{2}\right)}.$$

In consequence of the definitions I–III and our selection of allied functions we have

$$\vartheta_1\left(x + \frac{\pi}{2}\right) = \vartheta_2(x),$$

$$\vartheta_2\left(x + \frac{\pi}{2}\right) = -\vartheta_1(x),$$

$$\vartheta_3\left(x + \frac{\pi}{2}\right) = \vartheta(x),$$

$$\vartheta\left(x + \frac{\pi}{2}\right) = \vartheta_3(x),$$

$$\vartheta_1\left(x + \frac{i \log q}{2}\right) = -iq^{-n/4} e^{inx}\,\vartheta(x),$$

$$\vartheta\left(x + \frac{i \log q}{2}\right) = -iq^{-n/4} e^{inx}\,\vartheta_1(x);$$

where $n = 1$ when the ϕ, ψ are analytic. Then by substitution of $x + \pi/2$ for x in the last two equations we get for $n = 1$

$$\vartheta_2\left(x + \frac{i \log q}{2}\right) = q^{-1/4} e^{ix}\,\vartheta_3(x),$$

$$\vartheta_3\left(x + \frac{i \log q}{2}\right) = q^{-1/4} e^{ix}\,\vartheta_2(x).$$

For $i = 1, 3$ equation (37) gives

$$\vartheta_2^2(0)\,\vartheta^2(x) = \vartheta^2(0)\,\vartheta_2^2(x) + \vartheta_3^2(0)\,\vartheta_1^2(x), \tag{54}$$

$$\vartheta_2^2(0)\,\vartheta_3^2(x) = \vartheta_3^2(0)\,\vartheta_2^2(x) + \vartheta^2(0)\,\vartheta_1^2(x). \tag{55}$$

The substitution of (II) in (25) gives

$$\vartheta(0)\vartheta_2(0)\vartheta_3(0)\vartheta_1(2x) = 2\vartheta_1(x)\vartheta(x)\vartheta_2(x)\vartheta_3(x), \tag{56}$$

while from (4) we have

$$\vartheta^3(0)\vartheta(2x) = \vartheta^4(x) - \vartheta_1^4(x) \tag{57}$$

with corresponding formulas for $\vartheta_2(x)$ and $\vartheta_3(x)$.

These formulas apply irrespective of the analytic or non-analytic character of the allied solutions. For differentiation formulas see the close of § 9.

9. A FUNDAMENTAL LEMMA

Formulas (25) and (27) furnish the fundamental limit

$$\operatorname*{Lim}_{x=0} \frac{\phi(2x)}{\phi(x)} = 2 \tag{58}$$

and hence also

$$\operatorname*{Lim}_{x=0} \frac{R(2x)}{R(x)} = 2. \tag{59}$$

The proof of these formulas was based on the existence of a root of $\phi(x)$ not at the origin, which was established on the hypothesis that $\phi(x), \psi(x)$ exist over the complex plane. In the investigation of the real solutions the limit (59) is fundamental, and it will therefore be necessary to establish it without assuming the existence of the functions exterior to the real axis.

For this purpose replace x, y in (B) first by $x+a, x-a$, then by $x/2+a$, $x/2-a$, and divide the first resulting equation by the second. We obtain

$$\frac{R(2x)}{R(x)} = \frac{R^2(x+a) - R^2(x-a)}{R^2\left(\frac{x}{2}+a\right) - R^2\left(\frac{x}{2}-a\right)} \cdot \frac{1 - R^2\left(\frac{x}{2}+a\right)R^2\left(\frac{x}{2}-a\right)}{1 - R^2(x+a)R^2(x-a)}. \tag{60}$$

The limit of the second fraction on the right for $x = 0$ is 1, if a is any point for which $R(a)$ is finite. Now by (B) we have

$$\pm R(x \pm a)R(a) = \frac{R^2\left(\frac{x}{2} \pm a\right) - R^2\left(\frac{x}{2}\right)}{1 - R^2\left(\frac{x}{2} \pm a\right)R^2\left(\frac{x}{2}\right)},$$

and therefore

$$\frac{R^2(x+a) - R^2(x-a)}{R^2(x/2+a) - R^2(x/2-a)} = \frac{1}{R^2(a)} \cdot \frac{R^2\left(\frac{x}{2}+a\right) + R^2\left(\frac{x}{2}-a\right) - R^2\left(\frac{x}{2}\right)[\ \ ]}{1 - R^2\left(\frac{x}{2}\right)[\ \ ]},$$

in which the square brackets abbreviate expressions which need not be written down. If this value is substituted in the first fraction on the right of (60) and the limit is then sought for $x = 0$, we immediately obtain the limit (59) and hence also (58).

A consequence of (58) is that the zeros of $\phi(x)$ must all be simple when it is analytic. For, the expansion

$$\phi(x) = A_0 x + A_1 x^2 + \cdots$$

is consistent with (58) only if $A_0 \neq 0$. The zero at the origin is therefore of first order, whence it follows by (5) and (6) that all other zeros of $\phi(x)$ and $\psi(x)$ must be of first order.

Regarding the differentiability of $R(x)$ it may be noted that it is differentiable over the entire axis or plane on which its existence is assumed, provided that it is differentiable at the origin. For from (B) we have

$$\frac{R(x) - R(y)}{x - y} = \frac{R(x+y)\,[1 - R^2(x)R^2(y)]}{R(x) + R(y)} \cdot \frac{R(x-y)}{x-y}.$$

If now y approaches x as a limit, this gives

$$\frac{dR(x)}{dx} = \frac{R(2x)\,[1 - R^4(x)]}{2R(x)} \cdot \left(\frac{dR(x)}{dx}\right)_{x=0}.$$

Application to solution I gives with the aid of (56) and (57)

$$\frac{d\left(\frac{\vartheta_1(x)}{\vartheta x}\right)}{dx} = \frac{\vartheta^2(0)\,\vartheta_2(x)\,\vartheta_3(x)}{\vartheta_2(0)\,\vartheta_3(0)\,\vartheta^2(x)} \cdot \left(d\frac{\vartheta_1(x)}{\vartheta(x)}{dx}\right)_{x=0}.$$

Similar formulas hold for the derivatives of $\vartheta_1(x)/\vartheta_2(x)$ and $\vartheta_1(x)/\vartheta_3(x)$.

10. Determination of the real solutions

In determining the real solutions of (A) we shall first seek real solutions of

$$(B) \qquad R(x+y)\,R(x-y) = \frac{R^2(x) - R^2(y)}{1 - R^2(x)R^2(y)}.$$

Independently of (A) this equation may be considered on the hypothesis that $R(x)$ is one-valued and continuous on the real axis or over the complex plane, except at isolated points at which it becomes infinite. When $R(x) = \infty$, the value to be attributed to the right-hand member is, of course, the limiting value $-1/R^2(y)$. This member loses significance only when

$$R^2(x) = R^2(y) = \pm 1.$$

When for particular values of x or y either $R(x+y)$ or $R(x-y)$ becomes

infinite, their product must tend to a limiting value equal to that taken by the right-hand member of (B).

By simple considerations it appears that $R(x)$ is an odd function which vanishes at the origin.

If $R^2(x) = \pm 1$ for $x = \gamma$, we have by (B)

$$R(x+\gamma)R(x-\gamma) = \pm 1.$$

Hence when we place $y = x + 2\gamma$ in (B), it follows that 2γ is an infinity of $R(x)$. The points at which $R^2(x) = \pm 1$ are therefore isolated.

Suppose $x = \alpha$ to be a root of $R(x)$ and put $y = x + \alpha$ in (B). We get

$$R^2(x+\alpha) = R^2(x) \tag{61}$$

except possibly for values of x which make $R(2x+\alpha) = \infty$. But then by modifying slightly such an exceptional value and passing in (61) to this value of x as limit, it follows that this equation is still valid. Hence α is a period of $R^2(x)$. In consequence the zeros of $R(x)$ on the real axis or any line through the origin must be isolated, unless $R(x)$ vanishes identically along the line.

Similarly when $x = \beta$ is an infinity of $R(x)$, put $y = x + \beta$ in (B). We get

$$R^2(x+\beta) = \frac{1}{R^2(x)},$$

the equation holding by considerations of continuity even if $R(x+2\beta) = 0$. Thus 2β must be a period and a zero of $R^2(x)$.

Let β be any infinity so situated that no infinity lies within the segment $(0, \beta)$. The segment cannot contain a zero $x = \alpha$, for α is a period of $R^2(x)$, and hence there would be an infinity within the segment. For any particular x within the segment we have either

$$R(x+\beta) = \frac{1}{R(x)} \qquad \text{or} \qquad R(x+\beta) = -\frac{1}{R(x)}. \tag{62}$$

The sign of the right-hand member cannot change within the segment because of the continuity of the function. Furthermore, for $y = \beta$ equation (B) gives

$$R(x+\beta)R(x-\beta) = \frac{1}{R^2(x)},$$

from which it follows that the same sign must continue to hold throughout the entire line joining $x = 0$ and $x = \beta$. When $R(x)$ is supposed to exist over the entire complex plane the same conclusion holds over the plane because $R(x)$ is continuous except at isolated poles. Thus for given β one or the other of equations (62) holds independently of x.

Let α denote a primitive root of $R(x)$. For any particular x equation (61) gives either

$$R(x+\alpha) = R(x) \qquad \text{or} \qquad R(x+\alpha) = -R(x). \tag{63}$$

If $\alpha/2$ is an infinity of $R(x)$, the former equation is in force because of (62). If $\alpha/2$ is not an infinity, then we can apply reasoning exactly like that of the preceding paragraph to show that for each α one or the other of the two equations (63) is universally applicable.

Seek now to determine the real solutions of (B) by arbitrarily assuming values for $R(x)$ at two initial points x', $2x'$. Unless $R(x)$ is constant and therefore everywhere 0, we can take an initial x' for which $R^2(x') \neq 0, 1, \infty$. If necessary, make a very small alteration in x' so that also $R(2x') \neq \infty$. Then by putting $x = x'$ and successively $y = 2x', 3x', \cdots$ in (B) we determine the sequence $R[(n+1)x']$, $(n = 2, 3, \cdots)$, unless we come to a value $n = n'$ which makes $R^2[(n'+1)x']$ equal to $R^2(x')$ or to $1/R^2(x')$. When this happens, $R(n'x')$ is 0 or ∞ so that one of the equations (62) or (63) respectively comes into play and $n'x'$ is a period or half-period. A comparison of $R(n'+1)x'$ with $R(x')$ will show which of the two corresponding equations is to be selected. The equation then serves to carry on the sequence of values for $R(nx')$ indefinitely. Thus $R(x)$ is determined uniquely over the point set $\{nx'\}$ in terms of the two initial values.

When y is successively placed equal to $2x$ and $3x$ in (B), two equations are obtained between which $R(3x)$ may be eliminated. If x in the resulting equation is replaced by $x'/2$, the equation may be written:

$$R^4\left(\frac{x'}{2}\right) - R^2\left(\frac{x'}{2}\right)\left[\frac{2 + \dfrac{R(2x')}{R(x')}\left(1 - R^4(x')\right)}{R^2(x')}\right] + 1 = 0, \tag{64}$$

which serves to determine $R(x'/2)$. This equation must not admit as root $R^2(x'/2) = \pm 1$, for then on replacing x' by $x'/2^m$ $(m = 1, 2, \cdots)$ we would obtain $R^2(x'/2^m) = \pm 1$ in conflict with the limit (59). To exclude these values our initial values $R(x')$, $R(2x')$ must now be subjected to the restriction that they shall not be connected by the relation

$$R(2x') = \frac{-2R(x')}{1 \pm R^2(x')}. \tag{65}$$

When the initial values satisfy the equation

$$R(2x') = \frac{2R(x')}{1 \pm R^2(x')}, \tag{66}$$

we obtain from (64)

$$R^2(x') = \left[\frac{2R\left(\frac{x'}{2}\right)}{1 \pm R^2\left(\frac{x'}{2}\right)}\right]^2,$$

and only then. When therefore the initial values satisfy either relation of (66), they must continue to satisfy the *same* relation when x' is replaced by $x'/2^m$, since otherwise the impossible relation (65) will be encountered. With this understanding (65) will be avoided when the argument is successively halved.

Select now some one of the roots of (64) as the value of $R(x'/2)$ and do the same when x' is successively replaced by $x'/2^m$ $(m = 1, 2, \cdots)$. The choice of root is ultimately restricted by (59). Now the roots of (64) form two pairs, those of either pair being the negative of one another and the reciprocal of those of the other. Since $R(0) = 0$, the choice of roots is ultimately restricted to the smaller pair, and finally for sufficiently large m to a definite one of this pair because of (59). Let this be for $m \geqq N$. Then if the initial x' is replaced by $x'/2^N$, as we shall now suppose, and if the notation x' is retained for the new initial point, the determination of $R(x'/2^m)$ $(m = 1, 2, \cdots)$ will be unique.

By combination of preceding results $R(x)$ is determined uniquely in terms of the initial values over the dense set of points $\{nx'/2^m\}$, $(n, m, = 1, 2, \cdots)$. The result can be extended by the equation $R(-x) = -R(x)$ to the other half of the line connecting x' with the origin. It remains for subsequent consideration whether the set of values thus obtained is consistent with the continuity requirement for $R(x)$.

As yet the requirement of reality has not been imposed. Suppose henceforth the initial points and values to be real. Then $R(nx')$ is real, and it remains to examine under what conditions $R(x'/2^m)$ will be real. Clearly we obtain positive values for $R^2(x'/2)$ from (64) if and only if

$$\frac{R(2x')}{R(x')} \cdot [1 - R^4(x')] + 2 \geqq 2R(x')^2.$$

When $|R(x')| > 1$, this may be written

$$-\frac{R(2x')}{R(x')} \geqq \frac{2}{1 + R^2(x')} \qquad (|R(x')| > 1). \tag{67}$$

Thus the ratio

$$q(x') \equiv \frac{R(2x')}{R(x')}$$

should be negative for $|R(x')| > 1$.

If $|R(x')| < 1$, we have

$$q(x') \geqq -\frac{2}{1+R^2(x')} \qquad (|R(x')|<1). \tag{68}$$

Suppose henceforth that the initial values meet these conditions and seek whether they continue to fulfill them when x' is replaced by $x'/2$. Three cases will be distinguished, which with the transition cases cover all possibilities for the initial values.

Case 1.

$$|R(x')|<1, \qquad |q(x')|<\frac{2}{1+R^2(x')}.$$

If (64) is written

$$2+q(x')[1-R^4(x')] = q^2\left(\frac{x'}{2}\right)\left[1+R^4\left(\frac{x'}{2}\right)\right] \tag{69}$$

and is then combined with the second of the inequalities, we obtain

$$q^2\left(\frac{x'}{2}\right)\left[1+R^4\left(\frac{x'}{2}\right)\right]<4-2R^2\left(\frac{x'}{2}\right)q^2\left(\frac{x'}{2}\right),$$

whence follows

$$\left|q\left(\frac{x'}{2}\right)\right|<\frac{2}{1+R^2\left(\frac{x'}{2}\right)}.$$

Moreover, we must select for $R(x'/2)$ a root of (64) having an absolute value less than 1, since the condition (67) cannot be met when x' is replaced by $x'/2$. With this understanding both inequalities of case 1 will continue to be fulfilled when x' is replaced by $x'/2$.

Case 2.

$$\frac{2}{|1-R^2(x')|}>|q(x')|>\frac{2}{|1+R^2(x')|}.$$

First suppose $|R(x')|<1$. Then $q(x')$ is positive, as otherwise (68) is not fulfilled. We can now combine either the first or second inequality of case 2 with (69) in the same manner as in case 1 and thereby obtain like inequalities with $x'/2$ in place of x'.

Next suppose $|R(x')|>1$. Then $q(x')$ is negative, and (69) may be written:

$$2+|q(x')|\cdot[R^4(x')-1] = q^2\left(\frac{x'}{2}\right)\left[1+R^4\left(\frac{x'}{2}\right)\right].$$

The combination of this with the first inequality of case 2 gives in the same manner as before

$$\left|q\left(\frac{x'}{2}\right)\right|<\frac{2}{\left|1-R^2\left(\frac{x'}{2}\right)\right|}.$$

The other desired inequality for $|q(x'/2)|$ is obtained immediately by addition of

$$2q^2\left(\frac{x'}{2}\right)R^2\left(\frac{x'}{2}\right) \equiv 2R^2(x') > 2$$

and

$$q^2\left(\frac{x'}{2}\right)\left[1 + R^4\left(\frac{x'}{2}\right)\right] > 2q^2\left(\frac{x'}{2}\right)R^2\left(\frac{x'}{2}\right) > 2.$$

Thus the inequalities characteristic of case 2 continue to be satisfied when x' is replaced by $x'/2$.

The choice of root for $R(x'/2)$ in (64) is restricted by the requirement that $q(x'/2)$ shall be positive or negative according as $|R(x'/2)|$ is less or greater than 1.

Case 3.

$$|q(x')| > \frac{2}{|1 - R^2(x')|}.$$

This case may be handled in the same way as case 2.

In addition to these cases we have the following transition cases:

Case 4.

$$q(x') = \frac{2}{1 + R^2(x')}.$$

Case 5.

$$q(x') = \frac{2}{1 - R^2(x')}.$$

As already pointed out, these conditions are conserved when x' is replaced by $x'/2$ and they are realized by the hyperbolic and trigonometric tangents of cx.

The conditions characteristic of the several cases have thus been shown to be conserved when x' is replaced by $x'/2^m$, and conversely they must be conserved when x' is replaced by $2^m x'$. We will now prove that cases 1–3 for given initial values of $R(x')$, $R(2x')$ can be realized through our Jacobian quotients.

Consider first the solution (I). By (54)–(57) we have

$$R(2x) \equiv \frac{\vartheta_1(2x)}{\vartheta(2x)} = \frac{2\dfrac{\vartheta_1(x)}{\vartheta(x)}}{1 - \dfrac{\vartheta_1^2(x)}{\vartheta^2(x)}} \cdot \frac{\vartheta(0)\vartheta(0)\vartheta_2(x)\vartheta_3(x)}{\vartheta_2(0)\vartheta_3(0)\vartheta(x)\vartheta(x)} \Big/ \left(1 + \frac{\vartheta_1^2(x)}{\vartheta^2(x)}\right)$$

(70)

$$= \frac{2R(x)}{1 - R^2(x)} \cdot \frac{\sqrt{\left(1 - \dfrac{R^2(x)}{k}\right)\left(1 - kR^2(x)\right)}}{1 + R^2(x)},$$

in which we have now introduced the constants

$$\sqrt{k} = \frac{\vartheta_2(0)}{\vartheta_3(0)}, \qquad \sqrt{k'} = \frac{\vartheta(0)}{\vartheta_3(0)} \qquad (k^2 + k'^2 = 1)$$

which we suppose real. The real function

$$R(x) \equiv \frac{\vartheta_1(x)}{\vartheta(x)} = \sqrt{k}\ \mathrm{sn}\ v \qquad (v = \vartheta_3^2(0)x)$$

is periodic and takes all values between $\sqrt{k}$ and $-\sqrt{k}$. If k is taken sufficiently large, there will be a set of points at which it takes an initial value $R(x')$ assigned under case 1. *By replacing x by cx* any one of these points may be made to coincide with the initial point x'. *The position of x' on the real axis is therefore immaterial and need not be taken account of further in this and succeeding cases.*

Put now $x = x'$ in (70) and then let k vary continuously from $|R(x')|$ to 1. Then the radical in (70) varies from 0 to $1 - R^2(x')$, and correspondingly $|q(x')|$ varies between the limits characteristic of case 1. Hence $|R(2x')|$ for given $R(x')$ can be made to take any assigned value consistent with case 1 by proper choice of k. At the same time $R(2x')$ can be given a positive or negative sign at pleasure since upon replacing x' by $-(x' + \pi)$ we have

$$R(-x' - \pi) = \frac{-\vartheta_1(x' + \pi)}{\vartheta(x' + \pi)} = \frac{\vartheta_1(x')}{\vartheta(x')} = R(x'),$$

$$R(-2x' - 2\pi) = -R(2x').$$

Thus any initial values consistent with case 1 can be realized through our ϑ-quotient.

The curve $y = \vartheta_1(x)/\vartheta(x)$ has the essential characteristics of the sine curve,* consisting of a series of congruent arches alternately above and below the real axes which have their maximum amplitudes at the mid points of the arches. For a given value of $R(x') = \vartheta_1(x')/\vartheta(x')$ there are two sets of points x' periodically spaced, and for each point of one set $R(2x')$ can be made to take the given initial value by varying k. Whether the point varies in position or is kept *in statu quo* by replacing x by cx and varying c and k simultaneously is immaterial. Now in case 1 when formula (64) was applied to obtain in succession the terms

$$(71) \qquad R\left(\frac{x'}{2}\right), \quad R\left(\frac{x'}{4}\right), \quad R\left(\frac{x'}{8}\right), \quad \cdots$$

just two values were possible for each member after the preceding ones had been fixed. These two values differed in sign but ultimately the sign in the sequence (71) becomes invariable because of the existence of the limit (59).

* Cf. Krause, *Theorie der elliptischen Funktionen*, pp. 68–69.

To every possible succession of signs in (71) consistent with this requirement there corresponds a point of one of the two periodically spaced sets of points above. Precisely as in the case of the sine function when $\sin x'$ is given, and for the same reason, every possible succession of signs in (71) is realizable by appropriately selecting x' from our two sets of points. It follows therefore that through the Jacobian quotient (I) case 1 can be completely realized, and no further restrictions upon (71) are necessary than those which have been already discovered.

For the solution II we obtain in like manner by placing

$$R(x) = \frac{\vartheta_1(x)}{\vartheta_2(x)},$$

$$R(2x) = \frac{2R(x)}{1 - R^2(x)} \cdot \frac{\vartheta_2(0)\,\vartheta_2(0)\,\vartheta(x)\,\vartheta_3(x)}{\vartheta(0)\,\vartheta_3(0)\,\vartheta_2(x)\,\vartheta_2(x)} \Big/ \frac{}{1 + R^2(x)}$$

$$= \frac{2R(x)}{1 - R^2(x)} \cdot \frac{\sqrt{(1 + k'R^2(x))\left(1 + \dfrac{R^2(x)}{k'}\right)}}{1 + R^2(x)}.$$

The zeros of $\vartheta_1(x)$ and $\vartheta_2(x)$ are simple and alternate with each other along the real axis. There must therefore be a set of points x' on the real axis for which our periodic quotient $R(x)$ assumes any assigned real value $R(x')$. If now for fixed $R(x)$ we allow k' to vary from 1 to 0 in the above equation, the radical will vary from $1 + R^2(x)$ to ∞. The limits for $|q(x')|$ are therefore the same as in case 3, and hence the initially preassigned value for $|R(2x')|$ can be attained simultaneously with $R(x')$. Now in case 3 the sign of $R(2x')$ must be such that $q(x') = R(2x')/R(x')$ is positive when $|R(x')| < 1$ and negative when $|R(x')| > 1$. From the theory of the ϑ-functions, it is easily verified that as x increases from 0 to π, the quotient $R(x) = \vartheta_1(x)/\vartheta_2(x)$ increases from 0 through $R(\pi/4) = 1$ to ∞, then from $-\infty$ through $R(3\pi/4) = -1$ to 0. Correspondingly $q(x)$ increases from 0 to $q(\pi/4) = \infty$ and thence through negative values to $q(\pi/2) = 0$, while from $\pi/2$ to 0 it passes through the same cycle of changes in opposite order. Hence the proper sign of $R(2x')$ can be realized. Thus any initial values for $R(x')$ and $R(2x')$ consistent with case 3 can be realized through the ϑ-quotient (II).

It remains to see whether every possible sequence of values (71) resulting from the initial values through the use of (64) are realizable through our ϑ-quotient. Now both in cases 2 and 3 there were two possible choices for any member $R(x'/2^m)$ of (71) after the preceding members have been fixed by use of (64), and these two differ in sign since the ratio $q(x'/2^m)$ of two

successive members of (71) is positive or negative according as $|R(x'/2^m)|$ is taken less or greater than 1. The graph of $R(x) = \vartheta_1(x)/\vartheta_2(x)$ has the same general character as that of $\tan x$ since $R(x + \pi/2) = -1/R(x)$. For a given value of $R(x')$ there is a periodic set of points $x' \pm n\pi$, for each of which the initial values can be realized by variation of k in the manner already explained. As in the case of $\tan x'$ and for the same reason all possible choices of sign in (71) can be realized subject to the condition that all terms shall ultimately be of same sign. Thus case 3 can be completely realized through the ϑ-quotient (II) and with no further restriction upon (71) than has been already laid down.

The solution III makes $R(x) = \vartheta_1(x)/\sqrt{i}\,\vartheta_3(x)$, and correspondingly we have

$$R(2x) = \frac{2R(x)}{1 + R^2(x)} \cdot \frac{\dfrac{\vartheta_3(0)\,\vartheta_3(0)\,\vartheta(x)\,\vartheta_2(x)}{\vartheta(0)\,\vartheta_2(0)\,\vartheta_3(x)\,\vartheta_3(x)}}{1 - R^2(x)}$$

$$= \frac{2R(x)}{1 + R^2(x)} \cdot \frac{\sqrt{\left(1 + \frac{ik'}{k} R^2(x)\right)\left(1 - \frac{ik}{k'} R^2(x)\right)}}{1 - R^2(x)}.$$

Real values for $R(x)$ along the real axis are no longer obtained if the Jacobian constants q, k, k' are real, but it will be now shown that if k, k' are conjugate imaginaries, there will be a line through the origin on which $R(x)$ is real. Place $k'/k = e^{i\phi}$ so as to make $R(2x)$ real when $R(x)$ is real. Then since

$$i(k/k' - k'/k) = 2 \sin \phi,$$

the radical will vary with ϕ between the limits $1 + R^2(x)$ and $1 - R^2(x)$. Thus the limits for $|q(x)|$ are those given by the inequalities of case 2.

The integrals

$$K = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}, \qquad K' = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k'^2 x^2)}}.$$

are conjugate imaginaries and the real part of K'/K is positive.* Then $q = e^{-\pi K/K'}$ is a complex imaginary with an absolute value less than 1 so that the usual series defining $\vartheta_1(x)$, $\vartheta_3(x)$ have a significance. If

$$\tau = iK'/K \equiv e^{i\delta},$$

we get by well-known formulas for transformation

* See for example Schwarz's *Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen*, p. 32.

$$\frac{1}{\sqrt{i}}\vartheta_1(x,\tau)=\frac{1}{i}\vartheta(x,\tau+1)=-\sqrt{i\tau'}\,e^{-i\tau'x^2/\pi}\,\vartheta_1(x\tau',-\tau'),$$

(72)

$$\vartheta_3(x,\tau)=\vartheta(x,\tau+1)=-\sqrt{i\tau'}\,e^{-i\tau'x^2/\pi}\,\vartheta_2(x\tau',-\tau'),$$

in which

$$\tau'=\frac{1}{\tau+1}=\frac{1}{2}-\frac{i\sin\delta}{2(1+\cos\delta)}.$$

Since $\sin\delta$ is positive, $q'=e^{-i\pi\tau'}$ *is a pure imaginary* having a modulus less than 1, and the ordinary series apply for $\vartheta_i(x\tau',-\tau')$ $(i=1,2)$ with q' in place of q. For real values of the argument $x\tau'$ these series are real save for a factor $\sqrt[4]{q'}$ common to both series. Hence the quotient

$$R(x)=\frac{\vartheta_1(x,\tau)}{\sqrt{i}\vartheta_3(x,\tau)}\equiv\frac{\vartheta_1(x\tau',-\tau')}{\vartheta_2(x\tau',-\tau')}$$

is real along the line through the origin on which $x\tau'$ is real. Now the zeros of $\vartheta_1(u,q)$ and $\vartheta_2(u,q)$ for real values of u and all values of $|q|<1$ alternate along the real axis, being situated at $u=n\pi$ and $u=(2n+1)\pi/2$. Case 2 can now be discussed exactly as case 3 with like conclusion.

In case 4 the relation

$$R(x)=\frac{2R\left(\frac{x}{2}\right)}{1+R^2\left(\frac{x}{2}\right)}$$

makes $|R(x')|<1$ and necessitates also that $R(x')$ and $R(x'/2)$ shall be of like sign, and therefore also all terms of (71). The determination of $R(x)$ is unique when $R(x')$ is given and is realized through $\pm\tanh cx$. Case 5 is very similar to cases 2 and 3 and is realized through $\pm\tan cx$.

It is noteworthy that all the real solutions are analytic, being expressible in terms of ϑ-functions or their degenerations.

From the real solutions of (B) those of (A) can be quickly obtained. We have only to effect *one* resolution of each solution of (B) into real components satisfying (A) and then to multiply these components by a common arbitrary factor of form (11), with or without change of sign of either component. For cases 1 and 3 a real resolution is afforded by (I) and (II), the characteristic constant q in the ϑ-functions and g_2, g_3 in the σ-functions being real. The real factor $e^{-2\eta_1\omega_1x^2}$ can also be dropped, being of form (11). In case 2 the constant q is unreal and the resolution offered in III or (72) is not real. But

$$\phi(x)=\frac{\vartheta_1(cx,iq)}{\vartheta_2(0,iq)},\qquad \psi(x)=\frac{\vartheta_2(cx,iq)}{\vartheta_2(0,iq)}\qquad (c\text{ and }q\text{ real})$$

and

$$\phi(x) = \sqrt{(e_2 - e_3)(e_2 - e_1)}\,\sigma(cx), \qquad \psi(x) = \sigma_2(cx)$$

afford real resolutions, it being understood that the constants g_2, g_3 of these σ-functions are real with negative discriminant so that the quasi-periods $2\omega_1$, $2\omega_3$ are conjugate imaginaries.

11. Solutions for the complex plane

Consider now equations (A), supposing the functions which they contain to be analytic over the complex plane and hence to be entire functions. The form of $\phi(x)$, $\psi(x)$ was ascertained in § 7 when their systems of zeros are simply periodic. We need therefore only consider the doubly periodic case. Let α, α' be a primitive pair of zeros. We will take the positive real axis through the one or the other of these points, so making our choice that the other zero is made thereby to have a negative coefficient for its imaginary part. By replacing x by cx the real zero can then be made equal to π. If the other zero is denoted by $i \log q$, the absolute value of q will be less than 1. As shown in § 9, the zeros are of first order. Form now in terms of q the familiar infinite products for $\vartheta_1(x)$, $\vartheta(x)$, $\vartheta_2(x)$, $\vartheta_3(x)$. Then we have in I–III three solutions $\phi_i(x)$, $\psi_i(x)$ whose zeros occupy the only possible positions for the three allied solutions. Any other entire function whose zeros have the same positions as those of $\phi_i(x)$ or $\psi_i(x)$ can differ from it only by a factor of form $e^{G(x)}$, in which $G(x)$ denotes an entire function. Consider now any pair of functions $e^{G_1(x)}\phi_i(x)$, $e^{G_2(x)}\psi_i(x)$, which give a solution of (A). In accordance with (16) their quotient $R(x)$ must admit 2π and $2i \log q$ as periods. But $\phi_i(x)/\psi_i(x)$ by construction possesses these periods, and consequently $e^{G_1(x)-G_2(x)}$ must be doubly periodic. Being also an analytic function without infinities, this exponential must be a constant. Now any solution of (B) ceases to be a solution when altered by a constant factor unless the factor is a fourth root of unity. Hence

$$e^{G_1(x)} = \omega_4\, e^{G_2(x)},$$

where ω_4 is a fourth root of unity. Thus our pair of functions differ from $\phi_i(x)$, $\psi_i(x)$, or an equivalent solution, only by a common factor which as such must be of form (11). Being also analytic, this factor is e^{cx^2}. *Except therefore for such a common factor, the components of any analytic solution of (A) with a doubly periodic system of zeros are expressible in one of the forms I–III in terms of ϑ-functions (with x replaced by cx), or as an equivalent solution.*

Pass now to the consideration of analytic solutions of (B). While to analytic solutions of (A) there correspond analytic solutions of (B), the converse is not evident without study. For the case of a doubly periodic system of zeros a comparison of any solution of (B) with a corresponding quotient of ϑ-func-

tions possessing the same zeros and infinities does indeed serve to establish their identity save for a constant factor, but a corresponding basis for a like conclusion in the simply periodic case is not apparent. We shall proceed as follows so as to cover simultaneously both cases independently of any consideration of (A).

An analytic solution of (B) is determined by its section along a line through the origin. It has been seen that the latter is fixed by the values of $R(x)$ at two initial points x' and $2x'$ upon the line. A unique function is not thereby determined owing to the choice of roots offered for $R(x/2)$ in (64), but the choice becomes definite when x is sufficiently near to the origin. Consider now any particular analytic solution $R(x)$ and seek to determine a function $f(x) = \vartheta_1(cx, q)/\vartheta(cx, q)$ which will agree with $R(x)$ over a set of points $2x'', x'', x''/2, \cdots$ and therefore over the line on which they lie and over the entire plane.

First substitute the given initial values $R(2x')$ and $R(x')$ in (70). The equation can then be solved for $k + 1/k$, giving two values of k with product 1. Take for k one of the two values not exceeding 1 in absolute value. The parameter q of the ϑ-functions is expressible as an ordinary power series in k whose coefficients are positive and have a sum equal to unity.* The series converges for $|k| \leqq 1$ and makes $|q| \leqq 1$. If $k = 1$ or -1, the functions $\tanh cx$ and $\tan cx$ respectively are to be substituted in place of $\vartheta_1(cx)/\vartheta(cx)$. Otherwise we have $|q| < 1$ so that the ordinary series for $\vartheta_1(x)$ and $\vartheta(x)$ as functions of x and q apply. In this manner q is determined so as to accord with the initial values.

When (70) is combined with (64) so as to eliminate $R(2x)$, we obtain an equation between $R(x)$ and $R(x/2)$ exactly like (70), except possibly for the sign of the radical, and containing the same value of $k + 1/k$. The value of q continues therefore to correspond when x' is replaced by $x'/2^m$. Consider now the sequence

$$R(2x'), \quad R(x'), \quad R(x'/2), \quad \cdots.$$

As already explained, each term after the first two is *uniquely* derivable by (64) from the two preceding, provided that x' is selected sufficiently near to the origin, as will be supposed. A like remark applies to a similar sequence formed for the solution $f(x) = \vartheta_1(x)/\vartheta(x)$. Now a sufficiently small vicinity of $x = 0$ will be built by $f(x)$, or the two above degenerate functions, with continuous 1–1 correspondence of points upon a vicinity of the origin of the f-plane. If m is sufficiently large, $R(x'/2^m)$ must fall in this vicinity and hence there is a point x'' such that $f(x'') = R(x'/2^m)$. By a proper substitution of cx for x in $f(x)$ the point x'' may be made to coincide with

* Cf. Schwarz, loc. cit., p. 54.

$x'/2^m$. If furthermore m is sufficiently large, x'' will lie so close to the origin that $f(2x'')/f(x'')$ and $R(2x'')/R(x'')$ in accordance with (59) will both be very nearly equal to 2. Since also $f(2x'')$ and $R(2x'')$ can both be obtained from $f(x'') = R(x'')$ by the same equation (70), we must have $f(2x'') = R(2x'')$. Agreeing then at the two points x'', $2x''$ the two functions must also agree over the set $x''/2$, $x''/4$, $\cdots$ and hence over the line through the origin containing these points and over the complex plane. *Thus any analytic solution of (B) is found to be identical with some function*

$$\vartheta_1(cx, q)/\vartheta(cx, q)$$

or with one of its degenerations.

In conclusion, we will make a few incomplete remarks regarding the solution of (B) when $R(x)$ is not analytic but merely fulfills the continuity assumptions of § 10. The solutions depend upon three parameters, which we assume to be the values of $R(x)$ at three initial points x', y', $x' - y'$ which do not lie on a common line through the origin. Assume arbitrarily these values, avoiding for convenience the values $0, \pm 1, \pm i, \infty$. Then by use of (B) the function $R(x)$ may be determined over the dense network

$$\pm \frac{nx'}{2^m} \pm \frac{ny'}{2^{m'}} \qquad (m, n, m', n' = 0, 1, 2, \cdots).$$

It does not, however, appear under what conditions the various sets of values thus obtained are consistent with the requirement of continuity. For a given solution its determination by specification of its values at three initial points will be unique, provided these values are sufficiently near to the origin. From a variety of loose considerations it seems altogether likely that all non-analytic solutions, so far as they do not fall under the somewhat trivial case of § 3, are merely affine distortions of analytic solutions, exactly as established in the simply periodic case. If, however, this is not the case, then there are other non-analytic solutions obeying the same general laws as the ϑ-functions.

UNIVERSITY OF WISCONSIN,
July, 1915

A SET OF FOUR INDEPENDENT POSTULATES FOR BOOLEAN ALGEBRAS*

BY

B. A. BERNSTEIN

In these Transactions, October, 1913, Sheffer† presented the following set of independent postulates for Boole's logic:

I. There are at least two distinct K-elements.

II. Whenever a and b are K-elements, $a|b$ is a K-element.

Definition. $a' = a|a$.

III. Whenever a and the indicated combinations of a are K-elements, $(a')' = a$.

IV. Whenever a, b, and the indicated combinations of a and b are K-elements, $a|(b|b') = a'$.

V. Whenever a, b, c, and the indicated combinations of a, b, and c are K-elements, $[a|(b|c)]' = (b'|a)|(c'|a)$.

This is the most economical set of postulates that has so far been proposed for Boolean algebras. Not only is the number of primitive propositions considerably smaller than that of the smallest set‡ of previous date—five instead of nine—but also the special elements "zero," the "whole," and the "negative" are all defined and their properties deduced. This economy Dr. Sheffer effected by basing the algebra on the powerful operation of "rejection," |.§ Choosing the primitive ideas that Dr. Sheffer chose, I give below a set of four independent postulates from which his five Postulates I–V are easily deduced.

New set of four postulates for Boolean logic

If we take *class* K and *operation* | as primitive ideas,‖ Boolean logic can be deduced from the following postulates:

* Presented to the Society, August 3, 1915.

† H. M. Sheffer: *A set of five independent postulates for Boolean algebras, with application to logical constants*, these Transactions, vol. 14 (1913), pp. 481–488.

‡ Professor E. V. Huntington's. See his *Sets of independent postulates for the algebra of logic*, these Transactions, vol. 5 (1904), pp. 288–309. See also the writer's *A complete set of postulates for the logic of classes, etc.*, University of California Publications in Mathematics, vol. 1, no. 4, pp. 87–96.

§ Interpreted concretely, the "reject" $a|b$ may be read: "that which is neither a nor b."

‖ The symbol "$=$" is not taken as a primitive idea. By $a = b$, we agree to mean that a and b can be interchanged.

P_1. *K contains at least two distinct elements.*

P_2. *If a, b are elements of K, $a|b$ is an element of K.*

DEFINITION 1. $a' = a|a$.

P_3. *If a, b, and the combinations indicated are elements of K,*

$$(b|a)|(b'|a) = a.$$

P_4. *If a, b, c, and the combinations indicated are elements of K,*

$$a'|(b'|c) = [(b|a')|(c'|a')]'.$$

The following system satisfies all the Postulates P_1–P_4, and thus proves these postulates to be consistent:

K = a class of two elements e_1, e_2; $e_i|e_j$ = the element given by the following "rejection" table,

|	e_1	e_2
e_1	e_2	e_1
e_2	e_1	e_1

THEOREM 1. $(a')' = a$.

For, in P_4, let $b = c = a$. Then, by Definition 1 and P_3,* the left member becomes

$$a'|(a'|a) = (a|a)|(a'|a) = a;$$

and, by P_3, the right member becomes

$$[(a|a')|(a'|a')]' = (a')'.$$

DEFINITION 2. $a'' = (a')'$.

THEOREM 2. $a|(b|b') = a'$.

For, by 1, P_4, 1, and P_3,†

$$a|(b|b') = a''|(b''|b') = [(b'|a'')|(b''|a'')]' = [(b'|a)|(b''|a)]' = a'.$$

THEOREM 3. $[a|(b|c)]' = (b'|a)|(c'|a)$.

For, by 1, P_4, 1, 1,†

$$[a|(b|c)]' = [a''|(b''|c)]' = [(b'|a'')|(c'|a'')]''$$
$$= [(b'|a)|(c'|a)]'' = (b'|a)|(c'|a).$$

Since Sheffer's Postulates I, II are identical with Postulates P_1, P_2, and since Postulates III, IV, V correspond respectively to Theorems 1, 2, 3,—all of Sheffer's postulates, and hence all of Boolean logic, can be deduced from Postulates P_1–P_4. Indeed, the two sets of postulates are equivalent; for propositions P_1–P_4 can also be derived from I–V, as may be easily verified.

* The use of P_1 and P_2 will not be indicated.

† The use of Definition 2 is not indicated.

Independence of the four postulates

The following systems $\bar{P}_1$–$\bar{P}_4$ are such that $\bar{P}_i$ contradicts none of the Postulates P_1–P_4 except P_i, thus showing the independence of P_i from the rest of the postulates.

$\bar{P}_1$. K = a class of one element e; $e|e = e$.

$\bar{P}_2$. K = a class of two elements e_1, e_2; $e_i|e_j$ is given by the following table:

	e_1	e_2
e_1	e_2	x
e_2	x	e_1

where x is not in K. P_2 is false for $a \neq b$.

$\bar{P}_3$. K = a class of two elements e_1, e_2; $e_i|e_j$ is given by the following table:

	e_1	e_2
e_1	e_1	e_1
e_2	e_2	e_1

Here P_3 is false for $a = e_1, b = e_2$.

$\bar{P}_4$. K = a class of two elements e_1, e_2; $e_i|e_j$ is given by the following table:

	e_1	e_2
e_1	e_1	e_2
e_2	e_1	e_2

Here P_4 is false for $a = e_1, c = e_2$.

University of California,
May, 1915

TRANSFORMATIONS OF SURFACES Ω*

(SECOND MEMOIR)

BY

LUTHER PFAHLER EISENHART

In a former memoir† we developed a theory of transformations K of conjugate systems with equal point invariants into systems of the same kind. Subsequently‡ we considered the case in which the lines joining corresponding points on two surfaces in the relation of a transformation K form a normal congruence. These surfaces, being of a particular kind, are called *surfaces C*. We have shown that there are transformations of the K type, denoted by K_m, which transform a surface C into surfaces C. The surfaces $\overline{S}$, orthogonal to these normal congruences, are the surfaces Ω discussed by Demoulin.§ We showed that there exist transformations A_m of $\overline{S}$ into surfaces $\overline{S}_1$, such that $\overline{S}$ and a surface $\overline{S}_1$ envelope a two-parameter family of spheres, and in the correspondence between $\overline{S}$ and $\overline{S}_1$ thus established lines of curvature correspond. In the present memoir we extend our investigations concerning these transformations A_m, and more particularly apply the results to certain types of surfaces Ω.

Transformations A_m belong to the general class of transformations of Ribaucour, and in the first part of the paper we put the equations in a form possessed by all transformations of Ribaucour.

Guichard discovered‖ a class of surfaces possessing the following characteristic property: If $\overline{S}$ is such a surface, there exists an associate surface $\overline{S}_1$ such that the lines of curvature on the two surfaces have the same spherical representation, and the principal radii of curvature ρ_1, ρ_2 and ρ_1', ρ_2' of the respective surfaces are in the relation

$$\rho_1 \rho_2' + \rho_2 \rho_1' = \text{const.} \neq 0.$$

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* Presented to the Society, December 28, 1915.

† These Transactions, vol. 15 (1914), pp. 397–430. This memoir will be referred to as M_1.

‡ These Transactions, vol. 16 (1915), pp. 275–310. This memoir will be referred to as M_2.

§ Comptes Rendus, vol. 153 (1911), pp. 590–593, 705–707.

‖ Comptes Rendus, vol. 130 (1900), p. 159.

Demoulin* remarked that these surfaces are surfaces Ω. We apply the above general results to these *surfaces of Guichard* and show that they admit transformations A_m into surfaces of Guichard.†

The circles orthogonal to two surfaces in the relation of a transformation A_m form a cyclic system. We call the planes of these circles the circle-planes of the transformation. Beginning with § 8, we consider the case where for two or more transformations A_m the circle-planes are the same for these transformations. We find that there are *special surfaces* Ω depending upon six arbitrary constants which admit, in general, three transformations A_m such that the circle-planes are the same, and these *complementary* transformations are known when a special surface is given.

We have shown in M_2 that the transformations D_m of isothermic surfaces, discovered by Darboux‡ and studied at length by Bianchi,§ are transformations A_m. Applying the foregoing results to isothermic surfaces we are led to the special isothermic surfaces studied by Darboux and Bianchi in the memoirs indicated.

In like manner these results when applied to surfaces of Guichard lead to the special surfaces of Guichard previously studied by the author.‖

In §§ 3 and 19 a study is made of the envelope of the circle-planes of complementary transformations. These surfaces are applicable to quadrics when $\overline{S}$ is a special isothermic surface or a special surface of Guichard.

In the closing section it is shown that for certain surfaces Ω the transformations A_m go in pairs, the circle-planes of any pair being the same. But the transformations cannot be found directly as in the case of the complementary transformations of special surfaces. The surfaces of Guichard are of this type.

1. Equations of a surface C

In accordance with the definition of a surface C, its cartesian coördinates, x, y, z, a complementary function t, and the function $x^2 + y^2 + z^2 - t^2$ are solutions of an equation of the form

$$\frac{\partial^2 \theta}{\partial u \partial v} + \frac{\partial \log \sqrt{\rho}}{\partial v}\frac{\partial \theta}{\partial u} + \frac{\partial \log \sqrt{\rho}}{\partial u}\frac{\partial \theta}{\partial v} = 0. \tag{1}$$

* L. c., p. 707.

† These transformations are the same as those found by the author formerly by a method not based upon the underlying transformations K_m, Annali di matematica, ser. 3, vol. 22 (1914), pp. 191–248.

‡ Annales de l'École Normale Supérieure, ser. 3, vol. 16 (1899).

§ Annali di matematica, ser. 3, vol. 11 (1905), pp. 93–158.

‖ Annali, l. c., pp. 235 et seq.

The fundamental functions E, F, G, D, D'' of C are given by*

$$E = \left(\frac{\partial t}{\partial u}\right)^2 + \frac{1}{\rho}, \qquad F = \frac{\partial t}{\partial u}\frac{\partial t}{\partial v}, \qquad G = \left(\frac{\partial t}{\partial v}\right)^2 + \frac{1}{\rho},$$

$$H^2 = EG - F^2 = \frac{1}{\rho}\left[\left(\frac{\partial t}{\partial u}\right)^2 + \left(\frac{\partial t}{\partial v}\right)^2\right] + \frac{1}{\rho^2}, \tag{2}$$

$$\frac{D}{H} = \frac{m_0(\lambda_0 - \rho\theta_0)}{T_0} - L, \qquad \frac{D''}{H} = \frac{m_0(\lambda_0 + \rho\theta_0)}{T_0} - M,$$

where

$$L = \frac{1}{H^2\rho}\left[\frac{\partial^2 t}{\partial u^2} + \frac{\partial t}{\partial u}\frac{\partial \log\sqrt{\rho}}{\partial u} - \frac{\partial t}{\partial v}\frac{\partial \log\sqrt{\rho}}{\partial v}\right],$$

$$M = \frac{1}{H^2\rho}\left[\frac{\partial^2 t}{\partial v^2} - \frac{\partial t}{\partial u}\frac{\partial \log\sqrt{\rho}}{\partial u} + \frac{\partial t}{\partial v}\frac{\partial \log\sqrt{\rho}}{\partial v}\right], \tag{3}$$

and

$$\frac{\partial T_0}{\partial u} = m_0(\lambda_0 - \rho\theta_0)\frac{\partial t}{\partial u}, \qquad \frac{\partial T_0}{\partial v} = m_0(\lambda_0 + \rho\theta_0)\frac{\partial t}{\partial v}, \tag{4}$$

θ_0 being the solution of (1) which determines a conjugate surface C_0 of C (cf. M_2, § 3), and λ_0 being determined by the equations

$$\frac{\partial \lambda_0}{\partial u} = -\rho\frac{\partial \theta_0}{\partial u}, \qquad \frac{\partial \lambda_0}{\partial v} = \rho\frac{\partial \theta_0}{\partial v}. \tag{5}$$

By means of the above equations and the expressions [41], it can be shown that the expressions (2) for D and D'' satisfy the Codazzi equations.† There remains the Gauss equation‡ to be satisfied, which is reducible in consequence of (2) to

$$\left[\frac{m_0^2(\lambda_0^2 - \rho^2\theta_0^2)}{T_0^2} - L\frac{m_0}{T_0}(\lambda_0 + \rho\theta_0) - M\frac{m_0}{T_0}(\lambda_0 - \rho\theta_0)\right]\left[\left(\frac{\partial t}{\partial u}\right)^2 + \left(\frac{\partial t}{\partial v}\right)^2 + \frac{1}{\rho}\right] = \frac{\partial^2 \log\sqrt{\rho}}{\partial u^2} + \frac{\partial^2 \log\sqrt{\rho}}{\partial v^2}. \tag{6}$$

Recapitulating these results, we have

THEOREM 1. *The analytical determination of surfaces C consists in finding five functions $\lambda_0, \theta_0, T_0, \rho$, and t satisfying equations* (4), (5), (6) *and*

$$\frac{\partial^2 t}{\partial u \partial v} + \frac{\partial \log\sqrt{\rho}}{\partial v}\frac{\partial t}{\partial u} + \frac{\partial \log\sqrt{\rho}}{\partial u}\frac{\partial t}{\partial v} = 0. \tag{7}$$

* Cf. [35], [36], [43], [46]. A reference of this sort is to equations of the memoir M_2.

† E., p. 156. A reference of this sort is to the author's *Differential Geometry*, Ginn and Co., Boston, 1909.

‡ E., p. 155.

2. Equations of transformations A_m

The lines joining corresponding points on a surface C and the conjugate surface C_0 form a normal congruence. The orthogonal surfaces are surfaces Ω by definition. Let $\overline{S}$ be one of these surfaces. In M_2, § 8, we established a transformation of $\overline{S}$ of the Ribaucour type. In the present section we give another form to the equations of such transformations A_m.

If $\overline{X}$, $\overline{Y}$, $\overline{Z}$; $\overline{X}_1$, $\overline{Y}_1$, $\overline{Z}_1$; $\overline{X}_2$, $\overline{Y}_2$, $\overline{Z}_2$ denote the direction-cosines of the normal to $\overline{S}$ and of the tangents to its lines of curvature, we have from [9], [12], and [53]

$$\overline{X} = \frac{1}{T_0}(a_0 X_1 + b_0 X_2 + w_0 X) \tag{8}$$

and

$$\overline{X}_1 = \sqrt{\rho}\left(\frac{\partial x}{\partial u} + \frac{\partial t}{\partial u}\overline{X}\right), \qquad \overline{X}_2 = \sqrt{\rho}\left(\frac{\partial x}{\partial v} + \frac{\partial t}{\partial v}\overline{X}\right), \tag{9}$$

where X, Y, Z; X_1, Y_1, Z_1; X_2, Y_2, Z_2 are the direction-cosines of the normal to C and of the bisectors of the angles between the parametric curves on C.

In consequence of [7] and (8), equations (9) may be replaced by

$$\overline{X}_1 = \sqrt{\rho}\left[\left(\sqrt{E}\cos\omega + \frac{a_0}{T_0}\frac{\partial t}{\partial u}\right)X_1 + \left(-\sqrt{E}\sin\omega + \frac{b_0}{T_0}\frac{\partial t}{\partial u}\right)X_2 + \frac{w_0}{T_0}\frac{\partial t}{\partial u}X\right],$$

$$\overline{X}_2 = \sqrt{\rho}\left[\left(\sqrt{G}\cos\omega + \frac{a_0}{T_0}\frac{\partial t}{\partial v}\right)X_1 + \left(\sqrt{G}\sin\omega + \frac{b_0}{T_0}\frac{\partial t}{\partial v}\right)X_2 + \frac{w_0}{T_0}\frac{\partial t}{\partial v}X\right],$$

where 2ω is one of the angles between the parametric curves on C.

By means of the expressions [42] and [46] for a_0, b_0, w_0, namely

$$a_0 = -\frac{T_0}{2\cos\omega}\left(\frac{1}{\sqrt{E}}\frac{\partial t}{\partial u} + \frac{1}{\sqrt{G}}\frac{\partial t}{\partial v}\right),$$

$$b_0 = \frac{T_0}{2\sin\omega}\left(\frac{1}{\sqrt{E}}\frac{\partial t}{\partial u} - \frac{1}{\sqrt{G}}\frac{\partial t}{\partial v}\right), \qquad w_0 = \frac{T_0}{H\rho}, \tag{10}$$

these are reducible to

$$\overline{X}_1 = \frac{1}{\sin 2\omega\sqrt{E\rho}}\left(\sin\omega X_1 - \cos\omega X_2 + \frac{1}{\sqrt{G}}\frac{\partial t}{\partial u}X\right),$$

$$\overline{X}_2 = \frac{1}{\sin 2\omega\sqrt{G\rho}}\left(\sin\omega X_1 + \cos\omega X_2 + \frac{1}{\sqrt{E}}\frac{\partial t}{\partial v}X\right). \tag{11}$$

From (8) and (11) we have

$$X_1 = \frac{a_0}{T_0}\bar{X} + \frac{1}{2\cos\omega}\left(\frac{1}{\sqrt{E\rho}}\bar{X}_1 + \frac{1}{\sqrt{G\rho}}\bar{X}_2\right),$$

$$X_2 = \frac{b_0}{T_0}\bar{X} + \frac{1}{2\sin\omega}\left(-\frac{1}{\sqrt{E\rho}}\bar{X}_1 + \frac{1}{\sqrt{G\rho}}\bar{X}_2\right), \tag{12}$$

$$X = \frac{w_0}{T_0}\bar{X} + \frac{1}{H\sqrt{\rho}}\left(\frac{\partial t}{\partial u}\bar{X}_1 + \frac{\partial t}{\partial v}\bar{X}_2\right).$$

If $\bar{S}_1$ denotes a transform of the surface $\bar{S}$ by a transformation A_{m_1}, as developed in M_2, § 8, the two surfaces $\bar{S}$ and $\bar{S}_1$ are the envelope of a two-parameter family of spheres. From [98] and [99] it follows that the coördinates, ξ, η, ζ, of the centers of these spheres are given by equations of the form

$$\xi = \bar{x} + (\sigma - t)\bar{X} = \bar{x}_1 + (\sigma - t)\bar{X}', \tag{13}$$

where $\bar{x}_1$, $\bar{y}_1$, $\bar{z}_1$ are the cartesian coördinates of $\bar{S}_1$; $\bar{X}'$, $\bar{Y}'$, $\bar{Z}'$, are the direction cosines of the normal to $\bar{S}_1$; and

$$\sigma = \frac{T_0\,\theta_1}{m_0\,\lambda_0(\theta_1 - \theta_{01})}. \tag{14}$$

In order to explain the terms in this expression, we remark that $\bar{S}_1$, being a surface Ω, is normal to the lines joining corresponding points on two surfaces C_1 and C_{10}. These surfaces C_1 and C_{10} are in the relations of transformations K_{m_1} with C and C_0 respectively, and the transformation functions are θ_1, λ_1 and θ_{01}, λ_{01} respectively.

If x_{10}, y_{10}, z_{10} and x_1, y_1, z_1 denote the cartesian coördinates of C_{10} and C_1 respectively, we have

$$\bar{X}' = \frac{\lambda_{10}\,m_0}{T_{10}}(x_{10} - x_1), \qquad \bar{X} = \frac{\lambda_0\,m_0}{T_0}(x_0 - x), \tag{15}$$

where T_{10} is the analogue of T_0.

Since C, C_0, C_1, C_{10} form a quatern of surfaces under the relation of transformations K_m, we have from [73], [76], [77], [79] the relations

$$\lambda_{ij}\,\theta_i = \theta_{ij}\,\lambda_i - \theta_j\,\lambda_i + \theta_i\,\lambda_j, \qquad \theta_{ij}\,\lambda_i + \theta_{ji}\,\lambda_j = 0, \tag{16}$$

$$\lambda_{ij}\,\theta_i + \lambda_{ji}\,\theta_j = 0,$$

and

$$\lambda_{ij}\,\theta_i\,x_{ij} = \theta_{ij}\,\lambda_i\,x - \lambda_i\,\theta_j\,x_i + \lambda_j\,\theta_i\,x_j, \tag{17}$$

for $i = 1$, $j = 0$. Also from [9]

$$x_i - x = \frac{1}{m_i\,\lambda_i}(a_i\,X_1 + b_i\,X_2 + w_i\,X) \tag{18}$$

for $i = 1$.

By means of these equations the expression (15) can be given the form

$$\bar{X}' = \frac{m_0}{\theta_1 T_{10}}[(\theta_{10}\lambda_1 + \lambda_0\theta_1)(x_1 - x) + \lambda_0\theta_1(x - x_0)]$$

$$(19) \qquad = -\frac{m_0}{m_1 T_{10}}\left[\left(\frac{\theta_{01} - \theta_1}{\lambda_1 m_1}\lambda_0 a_1 + \frac{\theta_1 a_0}{m_0}\right)X_1 + \left(\frac{\theta_{01} - \theta_1}{\lambda_1 m_1}\lambda_0 b_1 + \frac{\theta_1 b_0}{m_0}\right)X_2 + \left(\frac{\theta_{01} - \theta_1}{\lambda_1 m_1}\lambda_0 w_1 + \frac{\theta_1 w_0}{m_0}\right)X\right].$$

If the expressions (12) for X_1, X_2, and X be substituted in this equation and use be made of the identities

$$(20) \qquad a_0 \sin\omega - b_0\cos\omega + \frac{w_0}{\sqrt{G}}\frac{\partial t}{\partial u} = 0, \qquad a_0\sin\omega + b_0\cos\omega + \frac{w_0}{\sqrt{E}}\frac{\partial t}{\partial v} = 0,$$

which follow from [42], and of the relation [97], namely

$$(21) \qquad (T_{10} - T_0)\theta_1 + (\theta_{01} - \theta_1)m_0\lambda_0(t_1 - t) = 0,$$

t_1 being the complementary function for C_1, we have

$$(22) \qquad \bar{X}' = \bar{X} + \frac{m_0\lambda_0(\theta_1 - \theta_{01})}{T_{10}\theta_1\lambda_1 m_1}\left\{\left[\frac{\Phi_{10}}{T_0} + \lambda_1 m_1(t_1 - t)\right]\bar{X} + \frac{(a_1\sin\omega - b_1\cos\omega)\sqrt{G} + w_1\dfrac{\partial t}{\partial u}}{\sqrt{\rho H}}\bar{X}_1 + \frac{(a_1\sin\omega + b_1\cos\omega)\sqrt{E} + w_1\dfrac{\partial t}{\partial v}}{\sqrt{\rho H}}\bar{X}_2\right\},$$

where we use the abbreviation

$$(23) \qquad \Phi_{ij} = a_i a_j + b_i b_j + w_i w_j \qquad (i \neq j).$$

From (13) we have accordingly

$$(24) \qquad \bar{x}_1 - \bar{x} = (t - \sigma)(\bar{X}' - \bar{X}) = -\frac{1}{\sigma_1 m_1}(\eta_1\bar{X}_1 + \beta_1\bar{X}_2 + \mu_1\bar{X}),$$

where

$$(25) \qquad \sigma_1 = \frac{T_{10}\theta_1\lambda_1}{T_0\theta_1 + m_0\lambda_0(\theta_{01} - \theta_1)t'}, \qquad \eta_1 = \frac{1}{\sqrt{\rho}H}\left[(a_1\sin\omega - b_1\cos\omega)\sqrt{G} + w_1\frac{\partial t}{\partial u}\right],$$

$$\beta_1 = \frac{1}{\sqrt{\rho}H}\left[(a_1\sin\omega + b_1\cos\omega)\sqrt{E} + w_1\frac{\partial t}{\partial v}\right], \qquad \mu_1 = m_1\lambda_1(t_1 - t) + \frac{\Phi_{10}}{T_0}.$$

From [100] it follows that the radius R of the variable sphere whose envelope is $\bar{S}$ and $\bar{S}_1$ is given by

$$R = \frac{m_0 \lambda_0 t(\theta_{01} - \theta_1) + T_0 \theta_1}{m_0 \lambda_0 (\theta_{01} - \theta_1)}. \tag{26}$$

We consider for a moment a general transformation of Ribaucour of any surface. If the equations of the transformation are written in the form (24), the functions $\eta_1, \beta_1, \mu_1, \sigma_1$ and the function τ_1 defined by

$$2m_1 \tau_1 \sigma_1 = \eta_1^2 + \beta_1^2 + \mu_1^2 \tag{27}$$

satisfy the equations:

$$\begin{aligned}
&\frac{\partial \mu_1}{\partial u} = -\sqrt{\bar{E}}\,\frac{\eta_1}{\rho_1}, && \frac{\partial \mu_1}{\partial v} = -\sqrt{\bar{G}}\,\frac{\beta_1}{\rho_2},\\
&\frac{\partial \tau_1}{\partial u} = \sqrt{\bar{E}}\,\eta_1, && \frac{\partial \tau_1}{\partial v} = \sqrt{\bar{G}}\,\beta_1,\\
&\frac{\partial \beta_1}{\partial u} = \frac{\eta_1}{\sqrt{\bar{G}}}\frac{\partial \sqrt{\bar{E}}}{\partial v}, && \frac{\partial \eta_1}{\partial v} = \frac{\beta_1}{\sqrt{\bar{E}}}\frac{\partial \sqrt{\bar{G}}}{\partial u},
\end{aligned} \tag{28}$$

$$\begin{aligned}
&\frac{\partial \log \sigma_1}{\partial u} = -\sqrt{\bar{E}_1}\,\frac{\eta_1}{\tau_1}, && \frac{\partial \log \sigma_1}{\partial v} = \sqrt{\bar{G}_1}\,\frac{\beta_1}{\tau_1},\\
&\frac{\partial \eta_1}{\partial u} = -\frac{\beta_1}{\sqrt{\bar{G}}}\frac{\partial \sqrt{\bar{E}}}{\partial v} + \mu_1\frac{\sqrt{\bar{E}}}{\rho_1} + m_1\sigma_1(\sqrt{\bar{E}} - \sqrt{\bar{E}_1}),\\
&\frac{\partial \beta_1}{\partial v} = -\frac{\eta_1}{\sqrt{\bar{E}}}\frac{\partial \sqrt{\bar{G}}}{\partial u} + \mu_1\frac{\sqrt{\bar{G}}}{\rho_2} + m_1\sigma_1(\sqrt{\bar{G}} + \sqrt{\bar{G}_1}),
\end{aligned}$$

where $\bar{E}$, $\bar{G}$ and $\bar{E}_1$, $\bar{G}_1$ are the first fundamental coefficients of the given surface $\bar{S}$ and its transform $\bar{S}_1$, and ρ_1, ρ_2 are the principal radii of curvature of $\bar{S}$. Conversely, every set of functions satisfying these equations determines a transformation of Ribaucour.*

The ratio τ_1/μ_1 is equal to the radius of the variable sphere of the transformation. When $\bar{S}$ is a surface Ω and the transformation is of the type A_m, it follows from [93] that

$$m_0 \lambda_0 (\theta_{01} - \theta_1) + T_0 m_1 \lambda_1 (t_1 - t) + \Phi_{10} = 0. \tag{29}$$

Hence from (25), (26), and (29) we have

$$\begin{aligned}
\tau_1 &= -\frac{m_0 \lambda_0 (\theta_{01} - \theta_1) t + T_0 \theta_1}{T_0},\\
\mu_1 &= -\frac{m_0 \lambda_0 (\theta_{01} - \theta_1)}{T_0} = \frac{m_0 (\lambda_1 \theta_{10} + \lambda_0 \theta_1)}{T_0}.
\end{aligned} \tag{30}$$

* Annali, ser. 3, vol. 22 (1914), pp. 191–248.

From these expressions follows

$$\tau_1 = \mu_1 t - \theta_1, \tag{31}$$

and from (21) and (30)

$$\theta_1 T_{10} = T_0[\mu_1(t_1 - t) + \theta_1]. \tag{32}$$

Hence from the first of (25) we have

$$\sigma_1 = \frac{\lambda_1}{\tau_1}[\mu_1(t_1 - t) + \theta_1] = \frac{\lambda_1}{\tau_1}(\mu_1 t_1 - \tau_1). \tag{33}$$

From [54] we have that the fundamental functions for a surface $\bar{S}$ are expressible in the form

$$\begin{gathered}\sqrt{\bar{E}} = \left[1 - \frac{m_0 t}{T_0}(\lambda_0 - \rho\theta_0)\right]\frac{1}{\sqrt{\rho}}, \\ \sqrt{\bar{G}} = \left[1 - \frac{m_0 t}{T_0}(\lambda_0 + \rho\theta_0)\right]\frac{1}{\sqrt{\rho}}, \\ \frac{\sqrt{\bar{E}}}{\rho_1} = \frac{m_0(\lambda_0 - \rho\theta_0)}{T_0\sqrt{\rho}}, \qquad \frac{\sqrt{\bar{G}}}{\rho_2} = \frac{m_0(\lambda_0 + \rho\theta_0)}{T_0\sqrt{\rho}}, \\ \bar{D} = \bar{E}/\rho_1, \qquad \bar{D}'' = \bar{G}/\rho_2.\end{gathered} \tag{34}$$

When equation (31) is differentiated and use is made of equations (28) and (34), we obtain

$$\eta_1 = \sqrt{\rho}\left(\mu_1\frac{\partial t}{\partial u} - \frac{\partial\theta_1}{\partial u}\right), \qquad \beta_1 = \sqrt{\rho}\left(\mu_1\frac{\partial t}{\partial v} - \frac{\partial\theta_1}{\partial v}\right). \tag{35}$$

A function θ_1 determining a transformation K_{m_1} of C, and consequently a transformation A_{m_1} of $\bar{S}$, must satisfy the equations [65], namely

$$\begin{gathered}\frac{\partial\theta_1}{\partial u} + \sqrt{E}\left[\cos\omega\, a_1 - \sin\omega\, b_1 + (t_1 - t)\frac{m_1\lambda_1}{T_0}(\cos\omega\, a_0 - \sin\omega\, b_0)\right] = 0, \\ \frac{\partial\theta_1}{\partial v} + \sqrt{G}\left[\cos\omega\, a_1 + \sin\omega\, b_1 + (t_1 - t)\frac{m_1\lambda_1}{T_0}(\cos\omega\, a_0 + \sin\omega\, b_0)\right] = 0.\end{gathered} \tag{36}$$

When these expressions are substituted in (35) and also the expression for μ_1 from (25), we obtain

$$\begin{gathered}\eta_1 = \frac{\sqrt{\rho}}{T_0}\left[\Phi_{10}\frac{\partial t}{\partial u} + T_0\sqrt{E}(a_1\cos\omega - b_1\sin\omega)\right], \\ \beta_1 = \frac{\sqrt{\rho}}{T_0}\left[\Phi_{10}\frac{\partial t}{\partial v} + T_0\sqrt{G}(a_1\cos\omega + b_1\sin\omega)\right].\end{gathered} \tag{37}$$

These equations are consistent with (25) in consequence of (10).

We recall from M_2, § 6, that the complementary function t_1 of C_1 is given by

$$\begin{aligned}\frac{\partial t_i}{\partial u} &= \frac{\rho}{\lambda_i}\left[(t_i - t)\frac{\partial \theta_i}{\partial u} + \theta_i \frac{\partial t}{\partial u}\right],\\ \frac{\partial t_i}{\partial v} &= -\frac{\rho}{\lambda_i}\left[(t_i - t)\frac{\partial \theta_i}{\partial v} + \theta_i \frac{\partial t}{\partial v}\right],\end{aligned} \tag{38}$$

for $i = 1$. Making use of these equations and the preceding equations, we obtain from (33) by differentiation

$$\begin{aligned}\frac{\partial \sigma_1}{\partial u} &= \frac{\eta_1}{\sqrt{\rho}\tau_1^2}\left[\frac{\lambda_1 \theta_1 t_1 m_0(\lambda_0 - \rho\theta_0)}{T_0} - \lambda_1 \mu_1 t_1 + \rho\theta_1 \tau_1\right],\\ \frac{\partial \sigma_1}{\partial v} &= \frac{\beta_1}{\sqrt{\rho}\tau_1^2}\left[\frac{\lambda_1 \theta_1 t_1 m_0(\lambda_0 + \rho\theta_0)}{T_0} - \lambda_1 \mu_1 t_1 - \rho\theta_1 \tau_1\right].\end{aligned} \tag{39}$$

From (28), (35), and (39) we have

$$\begin{aligned}\sqrt{\bar{E}_1} &= \frac{1}{\sqrt{\rho}(\tau_1 - \mu_1 t_1)}\left[\frac{\theta_1 t_1 m_0(\lambda_0 - \rho\theta_0)}{T_0} - t_1 \mu_1 + \frac{\rho\theta_1 \tau_1}{\lambda_1}\right],\\ \sqrt{\bar{G}_1} &= \frac{-1}{\sqrt{\rho}(\tau_1 - \mu_1 t_1)}\left[\frac{\theta_1 t_1 m_0(\lambda_0 + \rho\theta_0)}{T_0} - t_1 \mu_1 - \frac{\rho\theta_1 \tau_1}{\lambda_1}\right].\end{aligned} \tag{40}$$

From these expressions and (34) we get

$$\begin{aligned}\sqrt{\bar{E}_1} - \sqrt{\bar{E}} &= \frac{\tau_1}{\lambda_1\sqrt{\rho}(\tau_1 - \mu_1 t_1)}\left[\lambda_1(t - t_1)\frac{m_0(\lambda_0 - \rho\theta_0)}{T_0} - \lambda_1 + \rho\theta_1\right],\\ \sqrt{\bar{G}_1} + \sqrt{\bar{G}} &= \frac{-\tau_1}{\lambda_1\sqrt{\rho}(\tau_1 - \mu_1 t_1)}\left[\lambda_1(t - t_1)\frac{m_0(\lambda_0 + \rho\theta_0)}{T_0} - \lambda_1 - \rho\theta_1\right].\end{aligned} \tag{41}$$

Hence the last two of equations (28) may be written

$$\begin{aligned}\frac{\partial \eta_1}{\partial u} &= -\frac{1}{\sqrt{\bar{G}}}\frac{\partial \sqrt{\bar{E}}}{\partial v}\beta_1 - \frac{m_1}{\sqrt{\rho}}(\lambda_1 - \rho\theta_1) + \Phi_{10}\frac{m_0(\lambda_0 - \rho\theta_0)}{\sqrt{\rho}T_0^2},\\ \frac{\partial \beta_1}{\partial v} &= -\frac{1}{\sqrt{\bar{E}}}\frac{\partial \sqrt{\bar{G}}}{\partial u}\eta_1 - \frac{m_1}{\sqrt{\rho}}(\lambda_1 + \rho\theta_1) + \Phi_{10}\frac{m_0(\lambda_0 + \rho\theta_0)}{\sqrt{\rho}T_0^2}.\end{aligned} \tag{42}$$

To these formulas we add

$$\frac{\Phi_{10}}{T_0} + \sqrt{\rho}\left(\eta_1 \frac{\partial t}{\partial u} + \beta_1 \frac{\partial t}{\partial v}\right) = H\rho w_1, \tag{42'}$$

which is obtained by solving the second and third of equations (25) for a_1 and b_1 and by substituting these and the expressions (10) for a_0, b_0, w_0 in (23) for $i = 1, j = 0$.

3. Envelope of the circle-plane of a transformation A_m

It is a well-known fact that the circles orthogonal to two surfaces which are in the relation of a transformation of Ribaucour form a cyclic system, that is they are normal to an infinity of surfaces. We call the plane of this circle the *circle-plane of the transformation* and in this section we derive certain results concerning the surface S_0 which is the envelope of these planes.

From (24) it follows that the direction cosines of the normal to this plane are proportional to

$$\beta_1 \bar{X}_1 - \eta_1 \bar{X}_2, \qquad \beta_1 \bar{Y}_1 - \eta_1 \bar{Y}_2, \qquad \beta_1 \bar{Z}_1 - \eta_1 \bar{Z}_2. \tag{43}$$

Consequently the coördinates x_0, y_0, z_0, of S_0 are given by

$$x_0 = \bar{x} + p(\eta_1 \bar{X}_1 + \beta_1 \bar{X}_2) + q\bar{X}, \tag{44}$$

where p and q are to be determined. The conditions necessary and sufficient that S_0 be the envelope are that

$$\sum (\beta_1 \bar{X}_1 - \eta_1 \bar{X}_2) \frac{\partial x_0}{\partial u} = 0, \qquad \sum (\beta_1 \bar{X}_1 - \eta_1 \bar{X}_2) \frac{\partial x_0}{\partial v} = 0. \tag{45}$$

The derivatives of $\bar{X}_1, \bar{X}_2, \bar{X}$ are given by

$$\begin{aligned}
&\frac{\partial \bar{X}_1}{\partial u} = -\frac{1}{\sqrt{\bar{G}}} \frac{\partial \sqrt{\bar{E}}}{\partial v} \bar{X}_2 + \frac{\sqrt{\bar{E}}}{\rho_1} \bar{X}, \qquad \frac{\partial \bar{X}_1}{\partial v} = \frac{1}{\sqrt{\bar{E}}} \frac{\partial \sqrt{\bar{G}}}{\partial u} \bar{X}_2, \\
&\frac{\partial \bar{X}_2}{\partial u} = \frac{1}{\sqrt{\bar{G}}} \frac{\partial \sqrt{\bar{E}}}{\partial v} \bar{X}_1, \qquad \frac{\partial \bar{X}_2}{\partial v} = -\frac{1}{\sqrt{\bar{E}}} \frac{\partial \sqrt{\bar{G}}}{\partial u} \bar{X}_1 + \frac{\sqrt{\bar{G}}}{\rho_2} \bar{X}, \\
&\frac{\partial \bar{X}}{\partial u} = -\frac{\sqrt{\bar{E}}}{\rho_1} \bar{X}_1, \qquad \frac{\partial \bar{X}}{\partial v} = -\frac{\sqrt{\bar{G}}}{\rho_2} \bar{X}_2.^*
\end{aligned} \tag{46}$$

By means of these equations and (28) we find that equations (45) are equivalent to

$$\begin{aligned}
p &= \frac{1}{\bar{r}} \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right), \\
q &= \frac{1}{\bar{r}} \left[\mu_1 \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right) + m_1 \sigma_1 \left(\frac{\sqrt{\bar{G}_1}}{\sqrt{\bar{G}}} + \frac{\sqrt{\bar{E}_1}}{\sqrt{\bar{E}}} \right) \right], \\
\bar{r} &= m_1 \sigma_1 \left[\frac{1}{\rho_1} \left(1 + \frac{\sqrt{\bar{G}_1}}{\sqrt{\bar{G}}} \right) - \frac{1}{\rho_2} \left(1 - \frac{\sqrt{\bar{E}_1}}{\sqrt{\bar{E}}} \right) \right].
\end{aligned} \tag{47}$$

* E., p. 157.

From (28) we have

$$\frac{\partial \tau_1}{\partial u} + \rho_1 \frac{\partial \mu_1}{\partial u} = 0, \qquad \frac{\partial \tau_1}{\partial v} + \rho_2 \frac{\partial \mu_1}{\partial v} = 0. \tag{48}$$

Since these equations become the Rodrigues equations,* when τ_1 and μ_1 are replaced by $\bar{x}, \bar{y}, \bar{z}$ and $\overline{X}, \overline{Y}, \overline{Z}$ respectively, it follows that τ_1 is a solution of the point equation of $\bar{S}$ and μ_1 of the tangential equation. Hence $\mu_1, \overline{X}, \overline{Y}, \overline{Z}$ are the tangential coördinates of a surface Σ_0 which corresponds with parallelism of tangent planes to $\bar{S}$ and with lines of curvature in correspondence. If ξ_0, η_0, ζ_0 are the cartesian coördinates of Σ_0, their expressions as given by the general equations for tangential coördinates† are reducible in consequence of (28) and (46) to the form

$$\xi_0 = \mu_1 \overline{X} + \eta_1 \overline{X}_1 + \beta_1 \overline{X}_2. \tag{49}$$

Making use of these results, we have from (44)

$$dx_0 = \overline{X} d\omega + \xi_0 \, dp, \tag{50}$$

where

$$\omega = q - p\mu_1 = \frac{m_1 \sigma_1}{\bar{r}} \left(\frac{\sqrt{\bar{G}_1}}{\sqrt{\bar{G}}} + \frac{\sqrt{\bar{E}_1}}{\sqrt{\bar{E}}} \right). \tag{50'}$$

If we put

$$\psi = \mu_1 \omega + \frac{p}{2}(\mu_1^2 + \eta_1^2 + \beta_1^2) + \tau_1, \tag{51}$$

the linear element of S_0 may be given the form

$$ds_0^2 = d\omega^2 + 2dpd\psi. \tag{52}$$

These results which are true for any cyclic system will be applied in § 19 to certain transformations A_m.

4. Inverse of a transformation A_m

It is evident that the relation of a transformation A_m is entirely reciprocal, and in the subsequent theory it will be desirable for us to know the functions by means of which $\bar{S}$ is given as a transform of $\bar{S}_1$.

The equation analogous to (24) is

$$\bar{x} - \bar{x}_1 = -\frac{1}{\sigma_1^{-1} m_1} (\eta_1^{-1} \overline{X}_1' + \beta_1^{-1} \overline{X}_2' + \mu_1^{-1} \overline{X}'),$$

where $\overline{X}', \overline{Y}', \overline{Z}'$; $\overline{X}_1', \overline{Y}_1', \overline{Z}_1'$; $\overline{X}_2', \overline{Y}_2', \overline{Z}_2'$ are the direction-cosines of the normal to $\bar{S}_1$ and of the tangents to its lines of curvature. For any trans-

* E., p. 122.

† E., p. 163

formation of Ribaucour, defined by (24), these direction cosines are given by the following expressions:*

$$\bar{X}_1' = \bar{X}_1 + \frac{\eta_1}{\tau_1}(\bar{x}_1 - \bar{x}), \qquad \bar{X}_2' = -\bar{X}_2 - \frac{\beta_1}{\tau_1}(\bar{x}_1 - \bar{x}),$$

$$\bar{X}' = \bar{X} + \frac{\mu_1}{\tau_1}(\bar{x}_1 - \bar{x}).$$

Substituting these expressions in the above equation and replacing $\bar{x}_1 - \bar{x}$ by its value from (24), we get an equation of the form

$$A\bar{X} + B\bar{X}_1 + C\bar{X}_2 = 0.$$

Since this equation is also true when the X's are replaced by the Y's and Z's, it follows that A, B, and C must be zero. This gives the three equations

$$\sigma_1\frac{\mu_1^{-1}}{\mu_1} + \sigma_1^{-1} = \sigma_1\frac{\eta_1^{-1}}{\eta_1} + \sigma_1^{-1} = -\sigma_1\frac{\beta_1^{-1}}{\beta_1} + \sigma_1^{-1} = \frac{\mu_1\mu_1^{-1} + \eta_1\eta_1^{-1} - \beta_1\beta_1^{-1}}{m_1\tau_1}.$$

From these equations and (27) follow

$$\eta_1^{-1} = \rho\eta_1, \qquad \beta_1^{-1} = -\rho\beta_1, \qquad \mu_1^{-1} = \rho\mu_1, \qquad \sigma_1^{-1} = \rho\sigma_1,$$

where ρ is a factor of proportionality to be determined.

We have remarked that the radius of the variable sphere is τ_1/μ_1. Hence $\tau_1^{-1} = \rho\tau_1$. If this value be substituted in equations analogous to the third and fourth of (28), we have in consequence of others of these equations that $\rho = c/\tau_1\sigma_1$, where c denotes a constant. Hence

$$\eta_1^{-1} = c\eta_1/\tau_1\sigma_1, \qquad \beta_1^{-1} = -c\beta_1/\tau_1\sigma_1, \qquad \mu_1^{-1} = c\mu_1/\tau_1\sigma_1,$$
$$\tau_1^{-1} = c/\sigma_1, \qquad \sigma_1^{-1} = c/\tau_1. \tag{53}$$

It is readily found that these values satisfy equations analogous to (28).

5. Surfaces of Guichard

It is our purpose to apply the preceding results to the surfaces of Guichard, as defined in the introduction. Calapso† made a study of these surfaces and determined their characterization. He found that there are two types which he called *surfaces of Guichard of the first and second kinds*. He showed that the fundamental functions of a surface of the first kind are given by

$$\sqrt{\bar{E}} = e^{\xi}\sinh\alpha, \qquad \sqrt{\bar{G}} = e^{\xi}\cosh\alpha, \qquad \bar{F} = 0, \qquad \bar{D}' = 0, \tag{54}$$
$$\bar{D} = e^{\xi}\sinh\alpha(\cosh\alpha + h\sinh\alpha),$$
$$\bar{D}'' = e^{\xi}\cosh\alpha(\sinh\alpha + h\cosh\alpha),$$

* Annali, l. c., p. 196.

† Annali di matematica, ser. 3, vol. 11 (1905), pp. 201 et seq.

where ξ, α, and h are functions satisfying

$$\frac{\partial h}{\partial u} = (h + \coth \alpha)\frac{\partial \xi}{\partial u}, \qquad \frac{\partial h}{\partial v} = (h + \tanh \alpha)\frac{\partial \xi}{\partial v},$$

$$\frac{\partial^2 \alpha}{\partial u^2} + \frac{\partial^2 \alpha}{\partial v^2} + \coth \alpha \frac{\partial^2 \xi}{\partial u^2} + \tanh \alpha \frac{\partial^2 \xi}{\partial v^2} - \operatorname{csch}^2 \alpha \frac{\partial \alpha}{\partial u}\frac{\partial \xi}{\partial u}$$

(55)

$$+ \operatorname{sech}^2 \alpha \frac{\partial \alpha}{\partial v}\frac{\partial \xi}{\partial v} + (\cosh \alpha + h \sinh \alpha)(\sinh \alpha + h \cosh \alpha) = 0,$$

$$\frac{\partial^2 \xi}{\partial u \partial v} = \frac{\partial \xi}{\partial u}\frac{\partial \xi}{\partial v} + \coth \alpha \frac{\partial \xi}{\partial u}\frac{\partial \alpha}{\partial v} + \tanh \alpha \frac{\partial \xi}{\partial v}\frac{\partial \alpha}{\partial u}.$$

Conversely every set of functions satisfying these equations determines such a surface. In what follows we consider only surfaces of the first kind, but analogous results are equally true for surfaces of the second kind.

The fundamental functions of the associate surface $\bar{S}'$ are of the same form as (54) in functions ξ', α', h', given by

$$e^{\xi'} = e^{-\xi}(1 - h^2),$$

(56)

$$\sinh \alpha' = \frac{1}{h^2 - 1}[\sinh \alpha (1 + h^2) + 2h \cosh \alpha],$$

$$\cosh \alpha' = \frac{1}{1 - h^2}[\cosh \alpha (1 + h^2) + 2h \sinh \alpha].$$

Comparing equations (34) and (54), we have

$$m_0(\lambda_0 - \rho\theta_0) = \sqrt{\rho} T_0 (\cosh \alpha + h \sinh \alpha),$$

$$m_0(\lambda_0 + \rho\theta_0) = \sqrt{\rho} T_0 (\sinh \alpha + h \cosh \alpha),$$

(57)

$$T_0 - m_0 t(\lambda_0 - \rho\theta_0) = T_0 \sqrt{\rho} e^{\xi} \sinh \alpha,$$

$$T_0 - m_0 t(\lambda_0 + \rho\theta_0) = T_0 \sqrt{\rho} e^{\xi} \cosh \alpha.$$

Eliminating $\lambda_0 - \rho\theta_0$ and $\lambda_0 + \rho\theta_0$ from these equations, we obtain two equations which are equivalent to

$$t = e^{\xi}/(1 - h), \qquad \sqrt{\rho} = (1 - h)e^{-(\xi + \alpha)}. \tag{58}$$

Substituting these values in (57), we find

$$\lambda_0 = \frac{1}{2}\frac{T_0}{m_0}(1 - h^2)e^{-\xi}, \qquad \theta_0 = -\frac{1}{2}\frac{T_0}{m_0}e^{\xi}. \tag{59}$$

With the aid of the first two of equations (57) the first two of equations (4)

are reducible to

$$\frac{\partial \log T_0}{\partial u} = \sqrt{\rho}\,(\cosh \alpha + h \sinh \alpha)\frac{\partial t}{\partial u},$$
$$\frac{\partial \log T_0}{\partial v} = \sqrt{\rho}\,(\sinh \alpha + h \cosh \alpha)\frac{\partial t}{\partial v}. \tag{60}$$

From (58) and (55) we have

$$\frac{\partial t}{\partial u} = e^{\xi+a}\frac{\operatorname{csch} \alpha}{(1-h)^2}\frac{\partial \xi}{\partial u}, \qquad \frac{\partial t}{\partial v} = e^{\xi+a}\frac{\operatorname{sech} \alpha}{(1-h)^2}\frac{\partial \xi}{\partial v}. \tag{61}$$

Substituting these values in (60) and making use of (55), we find

$$\frac{\partial \log T_0}{\partial u} = \frac{1}{1-h}\frac{\partial h}{\partial u}, \qquad \frac{\partial \log T_0}{\partial v} = \frac{1}{1-h}\frac{\partial h}{\partial v}. \tag{62}$$

Since λ_0, θ_0, T_0 in the transformation from C to C_0 are determined only to within a constant factor, it follows that in all generality equations (62) may be replaced by

$$T_0 = \frac{1}{1-h}. \tag{63}$$

Consequently equations (59) become

$$\lambda_0 = \frac{1}{2}\frac{1+h}{m_0}e^{-\xi}, \qquad \theta_0 = \frac{1}{2}\frac{e^{\xi}}{(h-1)m_0}. \tag{64}$$

It is readily shown that these functions satisfy equations (5).

From (58) and (55) follow also

$$\frac{\partial \log \sqrt{\rho}}{\partial u} = \frac{e^a}{h-1}\operatorname{csch} \alpha \frac{\partial \xi}{\partial u} - \frac{\partial \alpha}{\partial u},$$
$$\frac{\partial \log \sqrt{\rho}}{\partial v} = \frac{e^a}{h-1}\operatorname{sech} \alpha \frac{\partial \xi}{\partial v} - \frac{\partial \alpha}{\partial v}. \tag{65}$$

With the aid of these results it can be shown that t, given by (58), is a solution of equation (1).

From the general existence theorem concerning surfaces of Guichard, as given by Calapso, it follows that each set of functions ξ, α, h, satisfying equations (55), determine a surface C, for which t and ρ are given by (58), and the surface C_0 is determined by the functions λ_0, θ_0, T_0, given by (63) and (64).

6. Transformation of surfaces of Guichard

It is our purpose now to show that there are transformations A_m of surfaces of Guichard into surfaces of the same kind, and that these transformations

are equivalent to certain ones obtained by us* previously in another manner.

In the first place we observe that there must exist two functions λ_{10}, θ_{10} for the new surface whose expressions are of the form (64), namely

$$\lambda_{10} = \frac{1}{2}\frac{1+h_1}{m_0}e^{-\xi_1}, \qquad \theta_{10} = \frac{1}{2}\frac{e^{\xi_1}}{(h_1-1)m_0}. \tag{66}$$

When the values (64) and (66) are substituted in the first of equations (16) for $i = 1, j = 0$, we have

$$\left(e^{-\xi_1}(1+h_1) - (1+h)e^{-\xi}\right)\theta_1 = \lambda_1\left(\frac{e^{\xi_1}}{h_1-1} - \frac{e^{\xi}}{h-1}\right). \tag{67}$$

From the general theory of transformations K† it follows that the point equation for C_1 is that obtained from (1) when ρ is replaced by ρ_1, where

$$\rho_1 = \frac{\lambda_1^2}{\rho\theta_1^2}. \tag{68}$$

From (58) it follows that we must have for the surface C_1

$$\sqrt{\rho_1} = (1-h_1)e^{-(\xi_1+\alpha_1)}. \tag{69}$$

When this value and that for ρ from (58) are substituted in (68), we get

$$\lambda_1 = \theta_1(1-h_1)(1-h)e^{-(\xi_1+\alpha_1+\xi+\alpha)}. \tag{70}$$

Eliminating λ_1 from equations (67) and (70), we get

$$(1-e^{-(\alpha_1+\alpha)})(e^{\xi}-e^{\xi_1}) + (1+e^{-(\alpha_1+\alpha)})(h_1e^{\xi} - he^{\xi_1}) = 0.$$

If we introduce a function k_1 by the equation

$$e^{\alpha_1+\alpha} = \frac{1-k_1}{1+k_1}, \tag{71}$$

the above equation may be replaced by

$$h = k_1 - l_1e^{\xi}, \qquad h_1 = k_1 - l_1e^{\xi_1}, \tag{72}$$

l_1 being a function thus defined.

From (71) it follows that

$$\begin{aligned}\cosh\alpha_1(1-k_1^2) &= (1+k_1^2)\cosh\alpha + 2k_1\sinh\alpha,\\ \sinh\alpha_1(1-k_1^2) &= -2k_1\cosh\alpha - (1+k_1^2)\sinh\alpha,\end{aligned} \tag{73}$$

and thence

$$\begin{aligned}\cosh\alpha_1 + k_1\sinh\alpha_1 &= \cosh\alpha + k_1\sinh\alpha = \phi_1,\\ \sinh\alpha_1 + k_1\cosh\alpha_1 &= -(\sinh\alpha + k_1\cosh\alpha) = -\psi_1,\end{aligned} \tag{74}$$

* L. c.

† M_1, p. 412.

ϕ_1 and ψ_1 being thus defined. Hence (73) may be written

$$(75)\qquad \begin{aligned}(1-k_1^2)\cosh\alpha_1 &= \phi_1 + k_1\psi_1,\\ (1-k_1^2)\sinh\alpha_1 &= -(\psi_1 + k_1\phi_1).\end{aligned}$$

From (54) and (58) it follows that t_1 and the fundamental quantities $\bar{E}_1$ and $\bar{G}_1$ for $\bar{S}_1$ must be of the form

$$(76)\qquad \sqrt{\bar{E}_1} = e^{\xi_1}\sinh\alpha_1, \qquad \sqrt{\bar{G}_1} = e^{\xi_1}\cosh\alpha_1, \qquad t_1 = \frac{e^{\xi_1}}{1-h_1}.$$

When these values are substituted in (40), the resulting equations are reducible by means of (57), (58), (72), and (74) to the single equation

$$(77)\qquad l_1\theta_1(1-h) + \mu_1(1-k_1) = 0.$$

From this and (31) we obtain

$$(78)\qquad \theta_1(1-h) + \tau_1(1-k_1) = 0,$$

and consequently

$$(79)\qquad \mu_1 = l_1\tau_1,$$

that is, $1/l_1$ is the radius of the variable sphere.

When the above values of λ_1, τ_1, μ_1, and t_1 are substituted in (33), we get

$$(80)\qquad \sigma_1 = \tau_1 e^{-(\xi_1+\xi)}(1-k_1^2).$$

Hence for the type of transformations A_m now under discussion equations (28) are

$$\begin{aligned}
&\frac{\partial\sigma_1}{\partial u} = e^{-\xi}\eta_1(k_1\phi_1+\psi_1), \qquad \frac{\partial\sigma_1}{\partial v} = e^{-\xi}\beta_1(k_1\psi_1+\phi_1),\\
&\frac{\partial\tau_1}{\partial u} = e^{\xi}\eta_1\sinh\alpha, \qquad \frac{\partial\tau_1}{\partial v} = e^{\xi}\beta_1\cosh\alpha,\\
&\frac{\partial\mu_1}{\partial u} = -\eta_1(\cosh\alpha + h\sinh\alpha), \qquad \frac{\partial\mu_1}{\partial v} = -\beta_1(\sinh\alpha + h\cosh\alpha),\\
(81)\quad &\frac{\partial\eta_1}{\partial u} = -\beta_1\left(\tanh\alpha\frac{\partial\xi}{\partial v} + \frac{\partial\alpha}{\partial v}\right) + \mu_1(\cosh\alpha + h\sinh\alpha)\\
&\qquad\qquad + m_1\sigma_1 e^{\xi}\sinh\alpha + m_1\tau_1 e^{-\xi}(k_1\phi_1+\psi_1),\\
&\frac{\partial\eta_1}{\partial v} = \beta_1\left(\coth\alpha\frac{\partial\xi}{\partial u} + \frac{\partial\alpha}{\partial u}\right), \qquad \frac{\partial\beta_1}{\partial u} = \eta_1\left(\tanh\alpha\frac{\partial\xi}{\partial v} + \frac{\partial\alpha}{\partial v}\right),\\
&\frac{\partial\beta_1}{\partial v} = -\eta_1\left(\coth\alpha\frac{\partial\xi}{\partial u} + \frac{\partial\alpha}{\partial u}\right) + \mu_1(\sinh\alpha + h\cosh\alpha)\\
&\qquad\qquad + m_1\sigma_1 e^{\xi}\cosh\alpha + m_1\tau_1 e^{-\xi}(k_1\psi_1+\phi_1).
\end{aligned}$$

In consequence of (79) equations (72) are equivalent to

$$h = k_1 - e^{\xi}\frac{\mu_1}{\tau_1}, \qquad h_1 = k_1 - e^{\xi_1}\frac{\mu_1}{\tau_1}. \tag{82}$$

When these values of h and h_1 respectively are substituted in the first two of equations (55) and analogous equations in functions with the subscript 1, we get

$$\begin{aligned} \frac{\partial k_1}{\partial u} &= \phi_1\left(\operatorname{csch}\alpha\frac{\partial\xi}{\partial u} + e^{\xi}\frac{\eta_1}{\tau_1}\right), \\ \frac{\partial k_1}{\partial v} &= \psi_1\left(\operatorname{sech}\alpha\frac{\partial\xi}{\partial v} + e^{\xi}\frac{\beta_1}{\tau_1}\right), \end{aligned} \tag{83}$$

and

$$\begin{aligned} \frac{\partial\xi_1}{\partial u} &= \sinh\alpha_1\left(\operatorname{csch}\alpha\frac{\partial\xi}{\partial u} + (e^{\xi_1} - e^{\xi})\frac{\eta_1}{\tau_1}\right), \\ \frac{\partial\xi_1}{\partial v} &= -\cosh\alpha_1\left(\operatorname{sech}\alpha\frac{\partial\xi}{\partial v} + (e^{\xi_1} - e^{\xi})\frac{\beta_1}{\tau_1}\right). \end{aligned} \tag{84}$$

When the value of e^{ξ_1} from equation (80) is substituted in these equations, they are satisfied identically in consequence of the above equations.

When the expression for k_1 from (71) is substituted in (83), we get

$$\begin{aligned} \frac{\partial}{\partial u}(\alpha_1 + \alpha) &= -(\cosh\alpha_1 + \cosh\alpha)\left(\operatorname{csch}\alpha\frac{\partial\xi}{\partial u} - e^{\xi}\frac{\eta_1}{\tau_1}\right), \\ \frac{\partial}{\partial v}(\alpha_1 + \alpha) &= (\sinh\alpha_1 - \sinh\alpha)\left(\operatorname{sech}\alpha\frac{\partial\xi}{\partial v} - e^{\xi}\frac{\beta_1}{\tau_1}\right). \end{aligned}$$

It is readily shown that these values of α_1 and ξ_1 satisfy also equations analogous to the last two of (55). Hence we have

THEOREM 2. *If $\bar{S}$ is a surface of Guichard of the first kind, each set of functions σ_1, τ_1, μ_1, η_1, β_1 satisfying* (81) *and* (27), *where k_1, ϕ_1, and ψ_1 are given by* (74) *and* (82), *determines a transformation A_{m_1} from $\bar{S}$ into a surface of the same kind; the functions α_1, ξ_1, h_1, of $\bar{S}_1$ are given by* (73), (80), *and* (82).

Hence in addition to the constants m_1 there are three arbitrary constants of integration.

From (31), (58), (70), (71), (80), and (82), we get

$$\begin{aligned} \theta_1 &= \frac{\tau_1}{1-h}(k_1 - 1), \\ \lambda_1 &= -\frac{\sigma_1(1-h_1)}{1-k_1} = -\sigma_1 - \mu_1 e^{-\xi}(1 + k_1), \end{aligned} \tag{85}$$

as the functions determining the transformation of the surface C associated with $\bar{S}$.

7. The associate surface of Guichard

In M_2, § 10, we found that the transformation A_{m_1}, for which

$$\theta_1 = -\lambda_1 = 1, \qquad \theta_{10} = \lambda_0, \qquad \lambda_{10} = \theta_0, \qquad \theta_{01} = -\lambda_{01} = 1, \tag{86}$$

transforms a surface $\bar{S}$ into a surface $\bar{S}_1$ corresponding to $\bar{S}$ with parallelism of tangent planes, instead of the surfaces $\bar{S}$ and $\bar{S}_1$ being the envelope of a two-parameter family of spheres. We shall show that if $\bar{S}$ is a surface of Guichard the surface $\bar{S}_1$ so obtained is the associate surface referred to in the definition of surfaces of Guichard, as given in the introduction.

From [111] we have

$$T_{10} = T_0. \tag{87}$$

On the assumption that $\bar{S}_1$ is a surface of Guichard we have from (63) and a similar expression for T_{10}

$$h_1 = h. \tag{88}$$

Likewise from (64), (66), and (86) we have

$$e^{\xi_1} = (1 - h^2)\, e^{-\xi}. \tag{89}$$

From the first two of equations (55) and similar equations for $\bar{S}_1$ we have in consequence of (88)

$$(h + \coth \alpha_1)\frac{\partial \xi_1}{\partial u} = (h + \coth \alpha)\frac{\partial \xi}{\partial u},$$

$$(h + \tanh \alpha_1)\frac{\partial \xi_1}{\partial v} = (h + \tanh \alpha)\frac{\partial \xi}{\partial v}.$$

If the expression for e^{ξ_1} from (89) is substituted in these equations, it is found that they are equivalent to

$$\begin{aligned} \sinh \alpha_1 &= \frac{1}{h^2 - 1}\left[\sinh \alpha\,(1 + h^2) + 2h \cosh \alpha\right], \\ \cosh \alpha_1 &= \frac{-1}{h^2 - 1}\left[\cosh \alpha\,(1 + h^2) + 2h \sinh \alpha\right]. \end{aligned} \tag{90}$$

But these equations and (88) define the associate surface of $\bar{S}$, as Calapso has shown.* Hence our assumption has been justified and we have

Theorem 3. *A surface of Guichard and its associate are in the relation of the parallel transformation A_m, that is, the one for which $\theta_1 = -\lambda_1 = 1$.*

8. Transformations A_m with the same circle-planes

We begin the study now of surfaces Ω which admit several transformations A_m with the same circle-planes.

* L. c., p. 214.

From the expressions (43) of the direction-parameters of the circle-plane of a transformation A_m it follows that if a second transformation A_m of $\bar{S}$ has the same circle-planes as the original transformation, the corresponding functions β_2 and η_2 are proportional to β_1 and η_1 respectively. From equations (28) and similar ones for the second transformation of $\bar{S}$, we note that we must have

$$\frac{\partial(\tau_1, \tau_2)}{\partial(u, v)} = 0, \qquad \frac{\partial(\mu_1, \mu_2)}{\partial(u, v)} = 0.$$

Consequently τ_2 is a function of τ_1, and μ_2 is a function of μ_1. But as τ_1 and τ_2 are solutions of the point equation of $\bar{S}$, τ_2 is at most a linear function of τ_1. The same is true of μ_2 and μ_1, since they satisfy the tangential equation of $\bar{S}$. On account of the linear character of equations (28) we have in all generality

$$\tau_2 = \tau_1 + \tau', \qquad \mu_2 = \mu_1 + \mu', \qquad m_2 = m_1 + m', \tag{91}$$

where τ', μ', and m' are constants. It follows at once from (28) that

$$\eta_2 = \eta_1, \qquad \beta_2 = \beta_1. \tag{92}$$

From (31) we have also

$$\theta_2 - \theta_1 = \mu' t - \tau'. \tag{93}$$

This equation is consistent with the equations obtained by replacing the quantities in (92) by the expressions (35) and similar ones for η_2 and β_2.

An exception to the preceding results arises in the case when $\bar{S}$ is a surface of revolution. In this case C coincides with $\bar{S}$ and ρ is a function of a single variable, say u. Now the point equation is satisfied by any function of u and consequently we are not justified in saying that τ_2 is necessarily linear in τ_1. As a matter to fact it is an easy matter to construct a large number of surfaces of revolution in relations of transformations of Ribaucour to the given surface, and the meridian planes are the circle planes. Hence we exclude from future consideration the case when $\bar{S}$ is a surface of revolution. It is readily shown that this is the only case for which τ_1 is a function of u or v alone.

From equations (42) and analogous ones in η_2 and β_2 we get

$$\begin{aligned}(\Phi_{20} - \Phi_{10})\, m_0 \lambda_0 &= T_0^2 (m_2 \lambda_2 - m_1 \lambda_1), \\ (\Phi_{20} - \Phi_{10})\, m_0 \theta_0 &= T_0^2 (m_2 \theta_2 - m_1 \theta_1),\end{aligned} \tag{94}$$

where Φ_{20} is given by (23) for $i = 2, j = 0$. In like manner from (37) and

analogous equations we have in consequence of (94)

$$(a_2 - a_1)\cos\omega - (b_2 - b_1)\sin\omega = \frac{T_0}{m_0\lambda_0}(m_1\lambda_1 - m_2\lambda_2)\frac{1}{\sqrt{E}}\frac{\partial t}{\partial u},$$

$$(a_2 - a_1)\cos\omega + (b_2 - b_1)\sin\omega = \frac{T_0}{m_0\lambda_0}(m_1\lambda_1 - m_2\lambda_2)\frac{1}{\sqrt{G}}\frac{\partial t}{\partial v}. \tag{95}$$

From [94] and [15] we have

$$\frac{\partial\Phi_{i0}}{\partial u} = m_i(\lambda_i - \rho\theta_i)T_0\frac{\partial t}{\partial u} - m_0(\lambda_0 - \rho\theta_0)\sqrt{E}(a_i\cos\omega - b_i\sin\omega),$$

$$\frac{\partial\Phi_{i0}}{\partial v} = m_i(\lambda_i + \rho\theta_i)T_0\frac{\partial t}{\partial v} - m_0(\lambda_0 + \rho\theta_0)\sqrt{G}(a_i\cos\omega + b_i\sin\omega), \tag{96}$$

for $i = 1, 2$. Making use of these equations, we obtain from (94) by differentiation

$$\frac{\partial}{\partial u}(\Phi_{20} - \Phi_{10}) = 2(\Phi_{20} - \Phi_{10})\frac{m_0(\lambda_0 - \rho\theta_0)}{T_0}\frac{\partial t}{\partial u},$$

$$\frac{\partial}{\partial v}(\Phi_{20} - \Phi_{10}) = 2(\Phi_{20} - \Phi_{10})\frac{m_0(\lambda_0 + \rho\theta_0)}{T_0}\frac{\partial t}{\partial v}.$$

In consequence of (4) these equations are equivalent to

$$\Phi_{20} - \Phi_{10} = kT_0, \tag{97}$$

where k is a constant. Hence equations (94) become

$$m_2\theta_2 - m_1\theta_1 = km_0\theta_0, \qquad m_2\lambda_2 - m_1\lambda_1 = km_0\lambda_0. \tag{98}$$

If m_1 and m_2 are not equal, it follows from (93) and (98) that θ_1 is a linear function of θ_0 and t. If m_1 and m_2 are equal, θ_0 must be a linear function of t. Hence we have

THEOREM 4. *In order that two transformations A_{m_1} and A_{m_2} have the same circle-planes, it is necessary that θ_1 be a linear function of θ_0 and t; if $m_1 = m_2$, θ_0 must be a linear function of t.*

We postpone the consideration of the latter case to § 20.

9. When θ_1 is linear in θ_0 and t

Since θ_1 for a transformation K_m is determined only to within a constant factor, we take in all generality

$$\theta_1 = \theta_0 + c_1 t + d_1, \tag{99}$$

where c_1 and d_1 are constants. When this value is substituted in equations

(36), we get

$$\sqrt{E}\,(\cos\omega\, a_1 - \sin\omega\, b_1) = R_1\frac{\partial t}{\partial u} - \frac{\partial\theta_0}{\partial u},$$

(100)

$$\sqrt{G}\,(\cos\omega\, a_1 + \sin\omega\, b_1) = R_1\frac{\partial t}{\partial v} - \frac{\partial\theta_0}{\partial v},$$

where we have put

(101) $$R_1 = m_1\lambda_1(t_1 - t) - c_1.$$

From the equations

(102) $$\frac{\partial\lambda_i}{\partial u} = -\rho\frac{\partial\theta_i}{\partial u}, \qquad \frac{\partial\lambda_i}{\partial v} = \rho\frac{\partial\theta_i}{\partial v},$$

for $i = 1$, we have by integration

(103) $$\lambda_1 = \lambda_0 + c_1 r + e_1,$$

where e_1 is a constant and r is defined by

(104) $$\frac{\partial r}{\partial u} = -\rho\frac{\partial t}{\partial u}, \qquad \frac{\partial r}{\partial v} = \rho\frac{\partial t}{\partial v}.$$

Differentiating equation (101) and making use of equations (38) for $i = 1$, we obtain

(105) $$\frac{\partial R_1}{\partial u} = -m_1(\lambda_1 - \rho\theta_1)\frac{\partial t}{\partial u}, \qquad \frac{\partial R_1}{\partial v} = -m_1(\lambda_1 + \rho\theta_1)\frac{\partial t}{\partial v}.$$

If the above expressions for θ_1 and λ_1 be substituted in these equations, they can be integrated with the result

(106) $$R_1 = -\left(m_1\frac{T_0}{m_0} + Crt + Et + Dr + F\right),$$

where we have put

(107) $$C = m_1 c_1, \qquad E = m_1 e_1, \qquad D = m_1 d_1, \qquad F = m_1 f_1,$$

f_1 being the constant of integration.

We consider now the second transformation. We put

(108) $$\theta_2 = \theta_0 + c_2 t + d_2, \qquad \lambda_2 = \lambda_0 + c_2 r + e_2$$

without loss of generality. Hence from (98) and (107) we have

(109) $$C = m_2 c_2, \qquad E = m_2 e_2, \qquad D = m_2 d_2, \qquad m_2 - m_1 = km_0.$$

The equations analogous to (100) are

$$\sqrt{E}\,(\cos\omega\, a_2 - \sin\omega\, b_2) = R_2\frac{\partial t}{\partial u} - \frac{\partial\theta_0}{\partial u},$$

$$\sqrt{G}\,(\cos\omega\, a_2 + \sin\omega\, b_2) = R_2\frac{\partial t}{\partial v} - \frac{\partial\theta_0}{\partial v},$$

where

$$R_2 = m_2 \lambda_2 (t_2 - t) - c_2 .$$

In order that these equations and (100) be consistent with (95), we must have

$$R_2 - R_1 = \frac{T_0}{m_0}(m_1 - m_2). \tag{110}$$

In consequence of this equation, (107), and (109), it follows that in the expression for R_2 similar to (106) we must have the same constant F, that is we may write

$$F = m_2 f_2 . \tag{111}$$

Again from (95) we have with the aid of (10), (98), and (109) the first two of the following:

$$a_2 - a_1 = \frac{m_2 - m_1}{m_0} a_0 , \qquad b_2 - b_1 = \frac{m_2 - m_1}{m_0} b_0 ,$$
$$w_2 - w_1 = \frac{m_2 - m_1}{m_0} w_0 . \tag{112}$$

The last one follows when we express the condition that the functions a_i and b_i satisfy the fundamental equations [10] of a transformation K, namely

$$\begin{aligned}
\frac{\partial a_i}{\partial u} &= - m_i (\lambda_i - \rho\theta_i) \sqrt{E} \cos\omega + b_i A + w_i D/2\sqrt{E} \cos\omega , \\
\frac{\partial a_i}{\partial v} &= - m_i (\lambda_i + \rho\theta_i) \sqrt{G} \cos\omega - b_i B + w_i D''/2\sqrt{G} \cos\omega , \\
\frac{\partial b_i}{\partial u} &= m_i (\lambda_i - \rho\theta_i) \sqrt{E} \sin\omega - a_i A - w_i D/2\sqrt{E} \sin\omega , \\
\frac{\partial b_i}{\partial v} &= - m_i (\lambda_i + \rho\theta_i) \sqrt{G} \cos\omega + a_i B + w_i D''/2\sqrt{G} \sin\omega , \\
\frac{\partial w_i}{\partial u} &= - \frac{D}{2\sqrt{E}} \left(\frac{a_i}{\cos\omega} - \frac{b_i}{\sin\omega} \right) , \\
\frac{\partial w_i}{\partial v} &= - \frac{D''}{2\sqrt{G}} \left(\frac{a_i}{\cos\omega} + \frac{b_i}{\sin\omega} \right) ,
\end{aligned} \tag{113}$$

where

$$A = \sqrt{\frac{E}{G}} \frac{\partial \log\sqrt{\rho}}{\partial v} \sin 2\omega - \frac{\partial\omega}{\partial u} , \qquad B = \sqrt{\frac{G}{E}} \frac{\partial \log\sqrt{\rho}}{\partial u} \sin 2\omega - \frac{\partial\omega}{\partial v} . \tag{114}$$

We recall from M_2, § 6, that a set of functions a_i, b_i, w_i, θ_i, λ_i, t_i satisfying

(36), (38), (102), (113), and the relation

$$2m_i\theta_i\lambda_i - (a_i^2 + b_i^2 + w_i^2) + m_i^2\lambda_i^2(t_i - t)^2 = 0, \tag{115}$$

determine a transformation K_m of surfaces C and a transformation A_m of surfaces Ω.

10. Transformations A_m with θ_1 linear in θ_0 and t

We consider now the transformation for which θ_1 has the form (99). From equations (100) we have, in consequence of (10),

$$\begin{aligned} a_1 &= -R_1\frac{a_0}{T_0} - \frac{1}{2\cos\omega}\left(\frac{1}{\sqrt{E}}\frac{\partial\theta_0}{\partial u} + \frac{1}{\sqrt{G}}\frac{\partial\theta_0}{\partial v}\right), \\ b_1 &= -R_1\frac{b_0}{T_0} - \frac{1}{2\sin\omega}\left(-\frac{1}{\sqrt{E}}\frac{\partial\theta_0}{\partial u} + \frac{1}{\sqrt{G}}\frac{\partial\theta_0}{\partial v}\right). \end{aligned} \tag{116}$$

When these expressions are substituted in the first four of equations (113) for $i = 1$, the resulting equations reduce to the following two:

$$\begin{aligned} &D\left(w_1 + \frac{w_0}{T_0}R_1\right) + (\theta_0)_{11} \\ &\qquad - \frac{1}{\rho}\left[m_1(\lambda_1 - \rho\theta_1) + \frac{m_0R_1}{T_0}(\lambda_0 - \rho\theta_0)\right] = 0, \\ &D''\left(w_1 + \frac{w_0}{T_0}R_1\right) + (\theta_0)_{22} \\ &\qquad - \frac{1}{\rho}\left[m_1(\lambda_1 + \rho\theta_1) + \frac{m_0R_1}{T_0}(\lambda_0 + \rho\theta_0)\right] = 0, \end{aligned} \tag{117}$$

where we have put

$$\begin{aligned} (\theta_0)_{11} &= \frac{\partial^2\theta_0}{\partial u^2} - \left\{\begin{matrix}11\\1\end{matrix}\right\}\frac{\partial\theta_0}{\partial u} - \left\{\begin{matrix}11\\2\end{matrix}\right\}\frac{\partial\theta_0}{\partial v}, \\ (\theta_0)_{22} &= \frac{\partial^2\theta_0}{\partial v^2} - \left\{\begin{matrix}22\\1\end{matrix}\right\}\frac{\partial\theta_0}{\partial u} - \left\{\begin{matrix}22\\2\end{matrix}\right\}\frac{\partial\theta_0}{\partial v}, \end{aligned} \tag{118}$$

the Christoffel symbols $\left\{\begin{matrix}rs\\t\end{matrix}\right\}$ being formed with respect to the linear element of C; their expressions are given by M_1 (108).

In like manner the last two of equations (113) become

$$\begin{aligned} \frac{\partial w_1}{\partial u} &= -\frac{R_1}{T_0}\frac{\partial w_0}{\partial u} + \frac{D}{H^2}\left(G\frac{\partial\theta_0}{\partial u} - F\frac{\partial\theta_0}{\partial v}\right), \\ \frac{\partial w_1}{\partial v} &= -\frac{R_1}{T_0}\frac{\partial w_0}{\partial v} + \frac{D''}{H^2}\left(E\frac{\partial\theta_0}{\partial v} - F\frac{\partial\theta_0}{\partial u}\right). \end{aligned} \tag{119}$$

If the first of equations (117) be differentiated with respect to v and the second with respect to u, and the derivatives of w_1 be replaced by their ex-

pressions from (119), the resulting equations vanish identically because of the Codazzi and Gauss equations* for C. Hence it is only necessary to express the condition that the expressions for w_1 given by (117) shall be equal. This is

$$\rho[D''(\theta_0)_{11} - D(\theta_0)_{22}] = D''\left[m_1(\lambda_1 - \rho\theta_1) + \frac{R_1}{T_0}m_0(\lambda_0 - \rho\theta_0)\right] - D\left[m_1(\lambda_1 + \rho\theta_1) + \frac{R_1}{T_0}m_0(\lambda_0 + \rho\theta_0)\right]. \tag{120}$$

From (101) and (106) we have

$$m_1\lambda_1(t_1 - t) = c_1 - \left(\frac{T_0}{m_0}m_1 + Crt + Et + Dr + F\right). \tag{121}$$

It is readily shown that this value of t_1 satisfies equations (38) for $i = 1$.

It can be shown that if a_i, b_i, w_i, λ_i, θ_i, t_i satisfy equations (36), (38), (102), and (113), the left-hand member of equation (115) is constant. As we desire a transformation K_m, we must show that for the above functions this constant is zero. In order to do this we must simplify the expression for w_1.

By definition the differential parameter $\Delta_1(\theta_0, t)$ formed with respect to the linear element of any surface is given by

$$\Delta_1(\theta_0, t) = \frac{G\dfrac{\partial\theta_0}{\partial u}\dfrac{\partial t}{\partial u} - F\left(\dfrac{\partial\theta_0}{\partial u}\dfrac{\partial t}{\partial v} + \dfrac{\partial\theta_0}{\partial v}\dfrac{\partial t}{\partial u}\right) + E\dfrac{\partial\theta_0}{\partial v}\dfrac{\partial t}{\partial v}}{EG - F^2}.$$

By differentiation we have

$$\begin{aligned}\frac{\partial}{\partial u}\Delta_1(\theta_0, t) &= \frac{G\dfrac{\partial t}{\partial u} - F\dfrac{\partial t}{\partial v}}{EG - F^2}(\theta_0)_{11} + \frac{G\dfrac{\partial\theta_0}{\partial u} - F\dfrac{\partial\theta_0}{\partial v}}{EG - F^2}(t)_{11},\\ \frac{\partial}{\partial v}\Delta_1(\theta_0, t) &= \frac{E\dfrac{\partial t}{\partial v} - F\dfrac{\partial t}{\partial u}}{EG - F^2}(\theta_0)_{22} + \frac{E\dfrac{\partial\theta_0}{\partial v} - F\dfrac{\partial\theta_0}{\partial u}}{EG - F^2}(t)_{22},\end{aligned} \tag{122}$$

where the expressions for $(t)_{11}$ and $(t)_{22}$ are analogous to those for $(\theta_0)_{11}$ and $(\theta_0)_{22}$ given by (118).

For a surface C, in consequence of (2), we have

$$\begin{aligned}G\frac{\partial t}{\partial u} - F\frac{\partial t}{\partial v} &= \frac{1}{\rho}\frac{\partial t}{\partial u}, \qquad E\frac{\partial t}{\partial v} - F\frac{\partial t}{\partial u} = \frac{1}{\rho}\frac{\partial t}{\partial v},\\ \Delta_1(\theta_0, t) &= \frac{\dfrac{\partial\theta_0}{\partial u}\dfrac{\partial t}{\partial u} + \dfrac{\partial\theta_0}{\partial v}\dfrac{\partial t}{\partial v}}{\rho H^2}.\end{aligned} \tag{123}$$

* E., p. 155.

Moreover, from [43],

$$(124)\qquad \begin{aligned}(t)_{11} &= -\frac{w_0 D}{T_0} + \frac{m_0}{\rho T_0}(\lambda_0 - \rho\theta_0),\\ (t)_{22} &= -\frac{w_0 D''}{T_0} + \frac{m_0}{\rho T_0}(\lambda_0 + \rho\theta_0).\end{aligned}$$

Hence for surfaces C we have from (122)

$$(125)\qquad \begin{aligned}\frac{1}{\rho H^2}\frac{\partial t}{\partial u}(\theta_0)_{11} &= \frac{\partial}{\partial u}\Delta_1(\theta_0, t) + \frac{G\dfrac{\partial\theta_0}{\partial u} - F\dfrac{\partial\theta_0}{\partial v}}{H^2 T_0}[w_0 D - m_0(\lambda_0 - \rho\theta_0)],\\ \frac{1}{\rho H^2}\frac{\partial t}{\partial v}(\theta_0)_{22} &= \frac{\partial}{\partial v}\Delta_1(\theta_0, t) + \frac{E\dfrac{\partial\theta_0}{\partial v} - F\dfrac{\partial\theta_0}{\partial u}}{H^2 T_0}[w_0 D'' - m_0(\lambda_0 + \rho\theta_0)].\end{aligned}$$

With the aid of these results, (117), and [48], namely

$$(126)\qquad \frac{\partial}{\partial u}\log w_0 = \frac{D}{H}\frac{\partial t}{\partial u}, \qquad \frac{\partial}{\partial v}\log w_0 = \frac{D''}{H}\frac{\partial t}{\partial v},$$

equations (119) may be written

$$(127)\qquad \begin{aligned}&\frac{\partial}{\partial u}\left[w_0\left(w_1 + \frac{R_1}{T_0}w_0\right)\right]\\ &\qquad = -T_0\frac{\partial}{\partial u}\Delta_1(\theta_0, t) + \frac{G\dfrac{\partial\theta_0}{\partial u} - F\dfrac{\partial\theta_0}{\partial v}}{H^2\rho}m_0(\lambda_0 - \rho\theta_0),\\ &\frac{\partial}{\partial v}\left[w_0\left(w_1 + \frac{R_1}{T_0}w_0\right)\right]\\ &\qquad = -T_0\frac{\partial}{\partial v}\Delta_1(\theta_0, t) + \frac{E\dfrac{\partial\theta_0}{\partial v} - F\dfrac{\partial\theta_0}{\partial u}}{H^2\rho}m_0(\lambda_0 + \rho\theta_0).\end{aligned}$$

In consequence of (4) and (123) these equations may be integrated in the form

$$(128)\qquad w_0\left(w_1 + \frac{R_1}{T_0}w_0\right) = m_0(\lambda_0\theta_0 + g) - T_0\Delta_1(\theta_0, t),$$

where g is a constant. From (110) and (112) it follows that g does not change in passing from the transformation of $\bar{S}$ into $\bar{S}_1$ to that of $\bar{S}$ into $\bar{S}_2$; the importance of this remark will be seen later.

11. Special surfaces Ω

When the values of a_1, b_1, w_1, t_1, as given by (116), (128), and (121) are substituted in equation (115) for $i = 1$, we get

$$\Delta_1\theta_0 + H^2\rho^2[\Delta_1(\theta_0, t) - N]^2 + 2C\left(Nrt + \frac{T_0}{m_0} - \theta_0 r - \lambda_0 t\right) + 2D(rN - \lambda_0) + 2E(tN - \theta_0) + 2FN + G = 0, \tag{129}$$

where

$$N = \frac{m_0(\lambda_0\theta_0 + g)}{T_0}, \qquad \Delta_1\theta_0 = \Delta_1(\theta_0, \theta_0), \tag{130}$$

and G is a constant such that

$$G = 2m_1 g + \frac{2CF}{m_1} - \frac{2DE}{m_1} - \frac{C^2}{m_1^2}. \tag{131}$$

When equation (129) is differentiated with respect to u and v separately, the resulting equations are reducible to*

$$\begin{aligned}\left(\frac{\partial\theta_0}{\partial u} - N\frac{\partial t}{\partial u}\right)\Big[(\theta_0)_{11} + \frac{1}{\rho}(CJ_1 + DK_1 + EL_1 + FM_1) + \frac{D}{w_0}T_0(N - \Delta_1(\theta_0, t))\Big] = 0,\\ \left(\frac{\partial\theta_0}{\partial v} - N\frac{\partial t}{\partial v}\right)\Big[(\theta_0)_{22} + \frac{1}{\rho}(CJ_2 + DK_2 + EL_2 + FM_2) + \frac{D''}{w_0}T_0(N - \Delta_1(\theta_0, t))\Big] = 0,\end{aligned} \tag{132}$$

where

$$\begin{aligned}M_i &= \frac{m_0(\lambda_0 \mp \rho\theta_0)}{T_0}, & J_i &= M_i rt - r \pm \rho t,\\ K_i &= M_i r \pm \rho, & L_i &= M_i t - 1,\end{aligned} \tag{133}$$

with the upper or lower sign according as i is 1 or 2. We consider now the various ways in which (132) can be satisfied.

If the first terms of (132) are zero, there is a functional relation between θ_0 and t. Since both are solutions of equation (1), this relation must be linear. Later (§ 17) we show that when θ_0 is a linear function of t, these terms are not equal to zero.

We consider next the case

$$\frac{\partial\theta_0}{\partial u} = N\frac{\partial t}{\partial u}, \qquad \frac{\partial\theta_0}{\partial v} \neq N\frac{\partial t}{\partial v},$$

$$(\theta_0)_{22} + \frac{1}{\rho}(CJ_2 + DK_2 + EL_2 + FM_2) + \frac{D''}{w_0}T_0(N - \Delta_1(\theta_0, t)) = 0. \tag{134}$$

* It should be noted that D which multiplies the last expression in the first equation is one of the fundamental functions for C and not a constant, as the other D is.

If the first of (134) be multiplied by $m_0(\lambda_0 - \rho\theta_0)$, it can be integrated, in consequence of (4), in the form

$$m_0(\lambda_0\theta_0 + g) = T_0 V, \tag{135}$$

that is, N is equal to V, which is a function of v alone.

Accordingly from the first of (134), (104), and (5) we have

$$\theta_0 = Vt + V_1, \qquad \lambda_0 = Vr + V_2, \tag{136}$$

where V_1 and V_2 are functions of v alone.

On the assumption that these quantities satisfy the second of (5) we get

$$V'r + V_2' = \rho(V't + V_1'),$$

where the primes indicate differentiation with respect to v. Since this value of r must satisfy equations (104), we must have

$$(V't + V_1')\sqrt{\rho} = V_3, \qquad 2\frac{\partial\sqrt{\rho}}{\partial v}\frac{V_3}{V'} + \rho\left(\frac{V_1'}{V'}\right)' - \left(\frac{V_2'}{V'}\right)' = 0, \tag{137}$$

where V_3 is a function of v given by

$$V_3^2 = V_1'V_2' - V'\left(\frac{V_1V_2 + g}{V}\right)'.$$

In order to obtain this expression for V_3, we solved the above equations for r and t, getting

$$r = \frac{1}{V'}(\sqrt{\rho}V_3 - V_2'), \qquad t = \frac{1}{V'}\left(\frac{V_3}{\sqrt{\rho}} - V_1'\right), \tag{138}$$

and substituted in the equation obtained by differentiating (135) with respect to v.

The remaining conditions to be satisfied are the last of (134) and (6). From the former of these we obtain an equation of the form

$$\frac{\partial\sqrt{\rho}}{\partial u} = P,$$

where P is a determinate function of V, V_1, V_2, their derivatives and $\sqrt{\rho}$, but the function is very complicated. However, in any particular case it would be possible to see whether it, (136), and (6) are consistent. Accordingly in what follows we make exception of the cases where θ_0 is of the form (136) or with V and V_1 replaced by U and U_1, functions of u alone.

When we equate to zero the second terms of the two equations (132), the resulting equations are reducible by means of (128) to (117). Hence, when C satisfies (129), the values of a_1, b_1, w_1, t_1, as given by (116), (128), and (121), determine a transformation K_{m_1} of C. Now m_1 is a solution of the cubic (131).

Moreover, each solution of this cubic determines a transformation of C into a surface C_i, and, as follows from § 9 the transformations A_m so determined have the same circle-planes.

From (18), (98), (109), and (112) we have

$$m_2 \lambda_2 x_2 - m_1 \lambda_1 x_1 + (m_2 - m_1) \lambda_0 x_0 = 0.$$

Hence if M_1, M_2, M_3 are the points on the three transforms of M in the above special case, it follows that M_1, M_2, M_3 are collinear and their line passes through M_0. We have seen (§ 8) that if m_2 and m_1 are equal, θ_0 must be a linear function of t. Hence for the present we exclude the case where two of the roots of (131) are equal (cf. § 20). This imposes a condition on the constants in equation (129), which will be assumed in what follows.

We shall say that two surfaces C whose functions satisfy the same equation (129) are *special surfaces C of the same class.* Later (§ 12) it will be shown that if C is a special surface, C_0 is a special surface of the same class. Hence we speak of $\bar{S}$ as a *special surface* Ω of the class determined by the constants $C, D, E, F, G,$ and g. Each of the three surfaces C_i determines a transform $\bar{S}_i$ of $\bar{S}$. We speak of these particular surfaces C_i and $\bar{S}_i$ as the *complementary transforms* of C and $\bar{S}$ respectively. The preceding results may be stated in

THEOREM 5. *If C is a special surface satisfying* (129) *in which the constants C, D, E, F, G, g are such that the roots of* (131) *are real and distinct, and θ_0 is not a linear function of t with coefficients which are functions of the same single variable or constants, there can be found directly three transforms of C such that corresponding points on these surfaces are collinear. These three surfaces C_i determine three transformations A_{m_i} of the surface $\bar{S}$, associated with C and C_0, such that the circle-planes of the three transformations coincide.*

The investigations of the last two sections enable us to state the following theorem which is fundamental in what follows:

THEOREM 6. *When a surface C admits a transformation K_m into another surface C_1, such that the function θ_1 is linear in t and θ_0 with constant coefficients, the surface C is a special surface, and C_1 is a complementary transform of C.*

12. COMPLEMENTARY TRANSFORMATIONS OF SPECIAL SURFACES Ω

Having thus established the existence of special surfaces Ω and transformations of them for which θ_1 is a linear function of t and θ_0, we consider in this section the surfaces $\bar{S}_i$ resulting from these transformations.

From (116) and (128) we get

$$\Phi_{10} = \sum a_1 a_0 = - R_1 T_0 + m_0 (\lambda_0 \theta_0 + g).$$

Hence from (25), (31), (33), and (35) follow, in consequence of (99) and (101),

$$\mu_1 = c_1 + N, \qquad \tau_1 = Nt - \theta_0 - d_1, \qquad \tau_1\sigma_1 = \mu_1\frac{(R_1 + c_1)}{m_1} + \lambda_1\theta_1,$$

(139)

$$\eta_1 = \sqrt{\rho}\left(N\frac{\partial t}{\partial u} - \frac{\partial\theta_0}{\partial u}\right), \qquad \beta_1 = \sqrt{\rho}\left(N\frac{\partial t}{\partial v} - \frac{\partial\theta_0}{\partial v}\right).$$

Equation [93] is reducible to

$$m_0\lambda_0(\theta_1 - \theta_{01}) = T_0(c_1 + N) = T_0\mu_1.$$

Hence in consequence of the identities (16), we have

$$\lambda_1\theta_{10} = -\lambda_0\theta_1 + \frac{T_0}{m_0}\mu_1, \qquad \theta_1\lambda_{10} = -\theta_0\lambda_1 + \frac{T_0}{m_0}\mu_1,$$

which are reducible to

$$\lambda_1\theta_{10} = -\lambda_0(c_1 t + d_1) + \frac{T_0}{m_0}c_1 + g,$$

(140)

$$\theta_1\lambda_{10} = -\theta_0(c_1 r + e_1) + \frac{T_0}{m_0}c_1 + g.$$

It is readily shown that these functions satisfy equations for $\bar{S}_1$ analogous to (5), namely

(141)

$$\frac{\partial\lambda_{10}}{\partial u} = -\frac{\lambda_1^2}{\rho\theta_1^2}\frac{\partial\theta_{10}}{\partial u}, \qquad \frac{\partial\lambda_{10}}{\partial v} = \frac{\lambda_1^2}{\rho\theta_1^2}\frac{\partial\theta_{10}}{\partial v}.$$

If θ_1^{-1} and λ_1^{-1} denote the functions by means of which we pass from $\bar{S}_1$ to $\bar{S}$, and if we show that

$$\theta_1^{-1} = \theta_{10} + c_1' t_1 + d_1',$$

then we shall have shown that $\bar{S}_1$ also is a special surface Ω.

We recall from M_2, § 1, that the fundamental equations of a transformation K of a surface C are

$$\frac{\partial}{\partial u}(\lambda_1 x_1) = -\rho\left(x\frac{\partial\theta_1}{\partial u} - \theta_1\frac{\partial x}{\partial u}\right),$$

(142)

$$\frac{\partial}{\partial v}(\lambda_1 x_1) = \rho\left(x\frac{\partial\theta_1}{\partial v} - \theta_1\frac{\partial x}{\partial v}\right).$$

For the inverse transformation from C_1 to C the analogous equations are

$$\frac{\partial}{\partial u}(\lambda_1^{-1}x) = -\frac{\lambda_1^2}{\rho\theta_1^2}\left(x_1\frac{\partial\theta_1^{-1}}{\partial u} - \theta^{-1}\frac{\partial x_1}{\partial u}\right),$$

$$\frac{\partial}{\partial v}(\lambda_1^{-1}x) = \frac{\lambda_1^2}{\rho\theta_1^2}\left(x_1\frac{\partial\theta_1^{-1}}{\partial v} - \theta_1^{-1}\frac{\partial x_1}{\partial v}\right).$$

From these equations it is readily found that

$$\theta_1^{-1} = \frac{\kappa}{\lambda_1}, \qquad \lambda_1^{-1} = \frac{\kappa}{\theta_1}, \tag{143}$$

where κ denotes an arbitrary constant.

When this value and the expressions for θ_{10} and t_1 from (140) and (121) respectively are substituted in the above equation, we find that it vanishes identically, if we take

$$c_1' = c_1, \qquad d_1' = d_1, \qquad \kappa = g - \frac{CF}{m_1^2} + \frac{DE}{m_1^2} + \frac{C^2}{m_1^3}. \tag{144}$$

Hence $\overline{S}_1$ is a special surface Ω. It remains to show that it is of the same class as $\overline{S}$.

We introduce for the surface C_1 a function r_1 analogous to r for C. From (104) and (141) it follows that

$$\frac{\partial r_1}{\partial u} = -\frac{\lambda_1^2}{\rho\theta_1^2}\frac{\partial t_1}{\partial u}, \qquad \frac{\partial r_1}{\partial v} = \frac{\lambda_1^2}{\rho\theta_1^2}\frac{\partial t_1}{\partial v}. \tag{145}$$

As thus defined, r_1 is determined only to within an additive constant. But it is readily shown that this constant may be so chosen that we have

$$\theta_1(r_1 - r) = \lambda_1(t_1 - t). \tag{146}$$

Since the functions λ_1^{-1} and θ_1^{-1} must satisfy equations (141), as well as λ_{10} and θ_{10}, we have

$$\theta_1^{-1} = \theta_{10} + c_1 t_1 + d_1, \qquad \lambda_1^{-1} = \lambda_{10} + c_1 r_1 + e_1, \tag{147}$$

in consequence of (144) and (146).

From these results it follows that the equation analogous to (121) is

$$m_1\lambda_1^{-1}(t - t_1) = c_1 - \left(\frac{T_{10}\, m_1}{m_0} + Cr_1 t_1 + Et_1 + Dr_1 + F'\right).$$

In consequence of (21), (16), (121), (140), (146), and (147) it follows that $F' = F$.

If the constant g is the same for $\overline{S}_1$ as for $\overline{S}$, the function μ_1^{-1} of the transformation from $\overline{S}_1$ to $\overline{S}$ must be

$$\mu_1^{-1} = c_1 + \frac{m_0}{T_{10}}(\lambda_{10}\theta_{10} + g),$$

as follows from (130) and (139). If μ_1^{-1} be replaced by its expression from (53), it is readily found that the preceding equation is true, when we take $c = \kappa$, as given by (144).

Since m_1 is the constant of the transformation from $\overline{S}_1$ to $\overline{S}$, from (131) it follows that G has the same value for $\overline{S}_1$ as for $\overline{S}$. Hence we have

THEOREM 7. *The complementary transforms of a special surface Ω are special surfaces of the same class.*

We consider now the four surfaces C, C_0, C_1, C_{10} which are associated with a pair of complementary surfaces $\bar{S}$ and $\bar{S}_1$. The functions θ_0^{-1} and λ_0^{-1}, by means of which C is obtained from C_0, are given by equations similar to (143), namely

$$\theta_0^{-1} = \kappa_0/\lambda_0, \qquad \lambda_0^{-1} = \kappa_0/\theta_0,$$

where κ_0 is a constant. From (140) and (16) we have

$$\theta_{01} = c_1 t + d_1 - \frac{T_0}{m_0 \lambda_0} c_1 - \frac{g}{\lambda_0}.$$

In consequence of [67], namely

$$T_0 = m_0 \lambda_0 (t - t_0), \tag{148}$$

it follows that if we take $\kappa_0 = -g$, we have

$$\theta_{01} = \theta_0^{-1} + c_1 t_0 + d_1. \tag{149}$$

On the assumption that r_0 is chosen so that a relation similar to (146) holds, we find from (16) and (149)

$$\lambda_{01} = \lambda_0^{-1} + c_1 r_0 + e_1. \tag{150}$$

For a *quatern* (cf. M_2, § 7) of surfaces C the functions t satisfy an equation similar to (17), namely

$$\lambda_{ij} \theta_i t_{ij} = \theta_{ij} \lambda_i t - \lambda_i \theta_j t_i + \lambda_j \theta_i t_j. \tag{151}$$

Also from (148) and an analogous equation we have

$$T_0^{-1} = m_0 \lambda_0^{-1} (t_0 - t) = -\lambda_0^{-1} T_0/\lambda_0.$$

With the aid of these equations it is readily shown that an equation analogous to (121), namely

$$m_1 \lambda_{01} (t_{10} - t_0) = c_1 - \left(\frac{T_0^{-1}}{m_0} m_1 + C r_0 t_0 + E t_0 + D r_0 + F \right), \tag{152}$$

is satisfied.

In accordance with the general theory of the permutability of transformations K_m, the constant m has the same value for the transformations from C into C_1 and C_0 into C_{10}. Hence all three constants m are the same for complementary transforms of C_0 as of C. In view of the above results g and G accordingly have the same values for C_0 as for C. Hence we have:

THEOREM 8. *When C is a special surface, the conjugate surface C_0, which with C determines a special surface Ω, is of the same class as C; if C_1 is a complementary transform of C, and C_{10} is the fourth surface of the quatern, then C_{10} is a special surface of the same class as C and is a complementary transform of C_0.*

13. General transformations of special surfaces

In this section we apply the results of the theorem of permutability of transformations A_m of surfaces Ω [§ 9] to show that if $\overline{S}$ and $\overline{S}_1$ are complementary surfaces Ω, it is possible to find surfaces $\overline{S}_2$ such that $\overline{S}_2$ and the surface $\overline{S}_{12}$ are complementary surfaces, where $\overline{S}_{12}$ is the surface forming with $\overline{S}$, $\overline{S}_1$, and $\overline{S}_2$ a quatern of surfaces Ω.

The function θ_{21} determining the transformation from $\overline{S}_2$ to $\overline{S}_{12}$ is given by [104], namely

$$\begin{aligned}\lambda_2\,\theta_{21}(m_2-m_1) &= m_2\lambda_2\,\theta_1 + m_1\lambda_1\,\theta_2 - \Phi_{12}\\ &\quad + m_1\,m_2\,\lambda_1\lambda_2\,(t_1-t)\,(t_2-t).\end{aligned} \tag{153}$$

From the general theory of transformations A_m we know that on the normal to $\overline{S}_2$ there are two points M_2 and M_{20} which describe a surface C_2 and a conjugate surface C_{20}, and that the four surfaces C, C_0, C_2, C_{20} form a quatern. From [93] we have that the function θ_{20}, determining the transformation from C_2 to C_{20}, is given by

$$\lambda_2\,\theta_{20} = -\lambda_0\,\theta_2 + \frac{T_0}{m_0}\,m_2\,\lambda_2\,(t_2-t) + \frac{\Phi_{20}}{m_0}. \tag{154}$$

When these expressions are substituted in

$$\theta_{21} = \theta_{20} + c_1\,t_2 + d_1, \tag{155}$$

we get

$$\begin{aligned}&(Cr+E+m_2\lambda_0)\,\theta_2 + (Ct_2+D+m_2\,\theta_0)\lambda_2 - \Phi_{12} + \frac{m_1-m_2}{m_0}\Phi_{20}\\ &\quad - m_2\lambda_2\,(t_2-t)\left(Crt+Dr+Et+F+m_2\frac{T_0}{m_0}\right) = 0.\end{aligned} \tag{156}$$

If the left-hand member of this equation be differentiated with respect to u and v separately, and it be required merely that θ_2 satisfy equations analogous to (36), the resulting equations vanish identically. Hence for all transformations A_{m_2} of $\overline{S}$ the left-hand member is a constant. Consequently of the ∞^r transformations A_{m_2} of $\overline{S}$ into $\overline{S}_2$, ∞^{r-1} are such that (155) is satisfied, and therefore, by Theorem 6, $\overline{S}_{12}$ is a complementary transform of $\overline{S}_2$. Hence $\overline{S}_2$ and $\overline{S}_{12}$ are special surfaces.

By methods analogous to those followed in the preceding section it can be shown that

$$\lambda_{21} = \lambda_{20} + c_1\,r_2 + e_1,$$

$$m_1\lambda_{21}\,(t_{21}-t_2) = c_1 - \left(T_{20}\frac{m_1}{m_0} + Cr_2\,t_2 + Et_2 + Dr_2 + F\right).$$

Analogously to (30) the function μ_{21} of the transformation from $\overline{S}_2$ to $\overline{S}_{12}$

is of the form

$$\mu_{21} = \frac{m_0 [\lambda_{21} (\theta_{21})_0 + \lambda_{20} \theta_{21}]}{T_{20}},$$

where $(\theta_{21})_0$ is the function determining the conjugate $(C_{21})_0$ of C_{21} such that C_2, C_{20}, C_{21}, $(C_{21})_0$ form a quatern. From equations analogous to [84] we have

$$(\theta_{21})_0 \lambda_{21} \theta_2 = -\theta_{20} \lambda_2 \theta_1 + \theta_{21} \lambda_2 \theta_0 + \theta_{10} \lambda_1 \theta_2 . \tag{157}$$

By means of these results we find that

$$\mu_{21} = c_1 + \frac{m_0 (\theta_{20} \lambda_{20} + g)}{T_{20}}.$$

As the transformation constant from C_2 to C_{12} is m_1, it follows that G is the same as for C. Hence:

THEOREM 9. *If $\bar{S}$ is a special surface Ω and $\bar{S}_1$ is a complementary transform, and $\bar{S}_2$ is obtained from $\bar{S}$ by a transformation A_{m_2} whose functions satisfy* (156), *$\bar{S}_2$ is a special surface of the same class, and $\bar{S}_{12}$, which is the fourth surface of the quatern determined by $\bar{S}$, $\bar{S}_1$, and $\bar{S}_2$, is a complementary transform of $\bar{S}_2$.*

If in the expression for Φ_{12}, as given by (23), we replace a_1, b_1, w_1, by their values from (116) and (128), then in consequence of (123), and equations analogous to (121), (42′), and (25), the equation (156) is reducible to

$$\begin{aligned} &(Cr + E + m_2 \lambda_0) \theta_2 + (Ct + D + m_2 \theta_0) \lambda_2 \\ &\quad + \sqrt{\rho} \left(\eta_2 \frac{\partial \theta_0}{\partial u} + \beta_2 \frac{\partial \theta_0}{\partial v} \right) - N \left[\frac{\Phi_{20}}{T_0} + \sqrt{\rho} \left(\eta_2 \frac{\partial t}{\partial u} + \beta_2 \frac{\partial t}{\partial v} \right) \right] \\ &\qquad - \mu_2 \left(Crt + Et + Dr + F + \frac{m_2 T_0}{m_0} \right) = 0 . \end{aligned} \tag{158}$$

From the preceding considerations we know that the left-hand member of this equation equated to a constant is a first integral of equations (28).

Since m_1 does not appear in equation (158), it follows that m_2 can take on the value m_1 as well as all other values. Hence we have

THEOREM 10. *When a special surface $\bar{S}$ is subjected to a transformation A_m whose functions satisfy equation* (158), *the new surface is a special surface of the same class.*

If C, C_i, C_j, C_{ij} form a quatern under general transformations K, the transformation function θ_{ij} is given by [72], namely

$$\begin{aligned} \frac{\partial}{\partial u} (\lambda_i \theta_{ij}) &= -\rho \left(\theta_i \frac{\partial \theta_j}{\partial u} - \theta_j \frac{\partial \theta_i}{\partial u} \right), \\ \frac{\partial}{\partial v} (\lambda_i \theta_{ij}) &= \rho \left(\theta_i \frac{\partial \theta_j}{\partial v} - \theta_j \frac{\partial \theta_i}{\partial v} \right). \end{aligned} \tag{159}$$

When θ_1 is of the form (99), and θ_2 is any other solution of equation (1), it is readily found that the expression (155) for θ_{21} satisfies these equations for $i = 2, j = 1$, if θ_{20} satisfies them for $i = 2, j = 0$. Hence if C and C_1 are complementary surfaces and C_2 is any transform of C, one of the infinity of surfaces S', which with C, C_1, C_2 forms a quatern under general transformations K, is determined by the value (155) of θ_{21}. But one, and only one, of the surfaces S' is a surface C (cf. [§ 9]), say C_{12}, in the relation of transformations K'_{m_2} and K'_{m_1} with C_1 and C_2 respectively. Moreover, we have seen that if the transformation K_{m_2} from C to C_2 satisfies (156), the function θ_{21} determining K'_{m_1} is of the form (155). Hence we have

THEOREM 11. *If C and C_1 are complementary surfaces for which $\theta_1 = \theta_0 + c_1 t + d_1$ and C_2 is any transform of C, the surface arising from C_2 by the transformation K determined by* (155) *is a surface C, if and only if* (156) *is satisfied; in this case it is complementary to C_2.*

14. THEOREM OF PERMUTABILITY OF GENERAL TRANSFORMATIONS OF SPECIAL SURFACES Ω

Let C_2 and C_3 be two surfaces obtained from a special surface C by transformations K_{m_2} and K_{m_3} which satisfy equations of the form (156), and let C_{23} be the unique surface C which with C, C_2, and C_3 forms a quatern of surfaces C. Let C_1 be the surface complementary to C determined by the constant m_1, and let C_{12} and C_{13} be the surfaces complementary to C_2 and C_3 in accordance with Theorem 11.

The functions θ_{21}, θ_{23}, θ_{31}, determining the transformations of C_2 into C_{12} and C_{23} and C_3 into C_{13}, are given by equations analogous to (153), namely

$$\lambda_i \theta_{ij}(m_i - m_j) = m_i \lambda_i \theta_j + m_j \lambda_j \theta_i - \Phi_{ij} + m_i m_j \lambda_i \lambda_j (t_i - t)(t_j - t) \qquad \begin{pmatrix} i = 2, 3, \\ j = 1, 3, \quad i \neq j \end{pmatrix}. \tag{160}$$

Also in accordance with (155)

$$\theta_{i1} = \theta_{i0} + c_1 t_i + d_1 \qquad (i = 2, 3). \tag{161}$$

From (23) and (18) it follows that

$$\Phi_{ij} = m_i m_j \lambda_i \lambda_j \sum (x_i - x)(x_j - x). \tag{162}$$

Hence equation (156) may be written

$$\begin{aligned}(Cr + E + m_2 \lambda_0)\theta_2 + (Ct + D + m_2 \theta_0)\lambda_2 \\ - m_1 m_2 \lambda_1 \lambda_2 \sum (x_1 - x)(x_2 - x) \\ + (m_1 - m_2) m_2 \lambda_0 \lambda_2 \sum (x_0 - x)(x_2 - x) \\ - m_2 \lambda_2 (t_2 - t)\left(Crt + Dr + Et + F + \frac{m_2 T_0}{m_0}\right) = 0.\end{aligned}$$

We desire to show that the transformation from C_2 to C_{23} satisfies a similar condition, namely

$$\begin{aligned}(Cr_2 + E + m_3\lambda_{20})\theta_{23} + (Ct_{23} + D + m_3\theta_{20})\lambda_{23} \\ - m_1 m_3 \lambda_{21}\lambda_{23}\sum (x_{12} - x_2)(x_{23} - x_2) \\ + (m_1 - m_3) m_3 \lambda_{20}\lambda_{23}\sum (x_{20} - x_2)(x_{23} - x_2) \\ - m_3\lambda_{23}(t_{23} - t_2)\left(Cr_2 t_2 + Dr_2 + Et_2 + F + m_3\frac{T_{20}}{m_0}\right) = 0.\end{aligned}$$

From equations (16) and (17) we have

$$(163)\quad \lambda_{2i}\theta_2(x_{2i} - x_2) = -(\lambda_i\theta_2 + \theta_{2i}\lambda_2)(x_2 - x) + \lambda_i\theta_2(x_i - x),$$

and similar equations in y, z, and t.

In consequence of the expression for T_{20} analogous to (21), and of equations (16) and (163), the above equation multiplied by θ_2^2 is reducible to

$$\begin{aligned}[C\theta_2 r + C\lambda_2(t_2 - t) + E\theta_2 + m_3(\lambda_2\theta_{20} - \lambda_2\theta_0 + \lambda_0\theta_2)]\theta_{23}\theta_2 \\ + [R(x_2 - x) - (x_3 - x)\lambda_3\theta_2] m_3 \{ m_1\lambda_1\theta_2(x_1 - x) \\ + (m_3 - m_1)\lambda_0\theta_2(x_0 - x) \\ - (x_2 - x)[m_3(\lambda_2\theta_{20} + \lambda_0\theta_2) + (Cr + E)\theta_2 + C\lambda_2 t_2 + D\lambda_2]\} \\ + (Ct_2 + D + m_3\theta_{20})\theta_2(R - \lambda_2\theta_3) \\ + m_3[R(t_2 - t) - (t_3 - t)\lambda_3\theta_2]\big((Ct_2 + D)(\theta_2 r + \lambda_2 t_2 - \lambda_2 t) \\ + (Et_2 + F)\theta_2 - \frac{C\theta_2}{m_3} + m_3(\lambda_0\theta_2 + \lambda_2\theta_{20})(t_2 - t) \\ - m_3\lambda_0\theta_2(t_0 - t)\big) = 0,\end{aligned}$$

where we have put

$$R = \theta_{23}\lambda_2 + \lambda_3\theta_2.$$

We replace the quantities $\sum (x_i - x)(x_j - x)$, for $i = 1, 2, 3; j = 1, 2, 3; i \neq j$, by their values from (162) and then Φ_{ij} by their values from (160). In like manner we replace $\sum (x_i - x)(x_0 - x)$ for $i = 2, 3$ by their values from (160) and then Φ_{i0} by expressions analogous to (154). Again analogously to [63], [64] we have

$$\sum (x_2 - x)^2 = \frac{2\theta_2}{m_2\lambda_2} + (t_2 - t)^2.$$

When these substitutions are made, we find that the equation is identically satisfied, in consequence of (99), (103), and (121). Hence the relation between C_2 and C_{23} is similar to that of C and C_2. It is evident that the same is true of the transformation from C_3 into C_{23}.

From the generalized theorem of permutability of transformations K [§ 7] it follows that there exists a unique surface S' which is in the relation of transformations K with C_{23}, C_{13}, and C_{12}. However, as yet we do not know that S' is a surface C; but presently this will be seen to be the case.

Now we shall show that θ'_{23}, the function determining the transformation from C_{23} to S', is given by

$$\theta'_{23} = (\theta_{23})_0 + c_1 t_{23} + d_1, \tag{164}$$

where t_{23} is the complementary function of C_{23}, and $(\theta_{23})_0$, given by an equation analogous to (157), is the function determining a conjugate surface to C_{23}.

From equations analogous to [84] we have

$$\theta'_{23}\lambda_{23}\theta_2 = -\theta_{21}\lambda_2\theta_3 + \theta_{23}\lambda_2\theta_1 + \theta_{31}\lambda_3\theta_2.$$

Substituting this expression and the expressions for $(\theta_{23})_0$ and t_{23} from equations of the form (157) and (163), we find that in consequence of (161) equation (164) is satisfied identically. Hence for the quatern C_2, C_{12}, C_{23}, S', the conditions of Theorem 11 are satisfied. Consequently S' is a surface C, complementary to C_{23} and we have

THEOREM 12. *If $\bar{S}$ is a special surface Ω and $\bar{S}_1$ and $\bar{S}_2$ are two special surfaces obtained by means of transformations A_{m_1} and A_{m_2} satisfying conditions of the form* (158), *the fourth surface of the quatern, determined by $\bar{S}$, $\bar{S}_1$, $\bar{S}_2$ in accordance with the theorem of permutability of transformations A_m, is a special surface of the same class.*

15. PARALLEL TRANSFORMATION OF SPECIAL SURFACES Ω

Let $\bar{S}$ be a special surface and $\bar{S}_1$ one of its complementary transforms. Also let $\bar{S}_2$ be the parallel transform of $\bar{S}$, as discussed in [§ 10]. In accordance with this theory there exists a fourth surface $\bar{S}_{12}$ such that $\bar{S}_1$ and $\bar{S}_{12}$ are in the relation of a parallel transformation. We desire to show that $\bar{S}_2$ and $\bar{S}_{12}$ are complementary transforms of one another and hence special surfaces Ω.

In analogy with (86) the transformation functions from $\bar{S}$ to $\bar{S}_2$ are given by

$$\theta_2 = -\lambda_2 = 1, \qquad \theta_{20} = \lambda_0, \qquad \lambda_{20} = \theta_0, \qquad \theta_{02} = -\lambda_{02} = 1. \tag{165}$$

The four surfaces C, C_0, C_2, and C_{20}, which are the basis of the transformation from $\bar{S}$ to $\bar{S}_2$ form a quatern such that C and C_2, and C_0 and C_{20} are pairs of associate surfaces. In a similar manner for the quatern C, C_1, C_2, C_{12}, C_1 and C_{12} are associate surfaces, as follows from [§ 10]. Hence we have

$$\theta_{21} = \lambda_1, \qquad \lambda_{21} = \theta_1, \qquad \theta_{12} = -\lambda_{12} = 1. \tag{166}$$

Now from (38) and an equation analogous to (146) we have

$$t_2 = r, \qquad r_2 = t. \tag{167}$$

Hence from the foregoing equations

$$\theta_{21} = \theta_{20} + c_1 t_2 + e_1, \qquad \lambda_{21} = \lambda_{20} + c_1 r_2 + d_1, \tag{168}$$

that is, the surfaces C_2 and C_{12} are complementary transforms of one another, and likewise $\overline{S}_2$ and $\overline{S}_{12}$. However we notice that the constants e_1 and d_1 are interchanged.

By methods analogous to those followed in § 13 it can be shown that the constants F, g, and G are the same for $\overline{S}$ and $\overline{S}_2$. Hence we have

THEOREM 13. *When a special surface Ω whose constants are C, D, E, F, G, g is subjected to the parallel transformation, the resulting surface is a special surface whose constants are C, E, D, F, G, g.*

16. SPECIAL ISOTHERMIC SURFACES

We apply the preceding results to the determination and transformation of special isothermic surfaces.*

In this case the lines of curvature being the conjugate system with equal point invariants, we have

$$E = G = e^{2\phi}, \qquad F = 0, \qquad \omega = 45°, \qquad \rho = e^{-2\phi}, \qquad H\rho = 1,$$

ϕ being thus defined. Since t is zero, we take $r = 0$ in conformity with (104), and have from (10) and (4),

$$t = r = 0, \qquad a_0 = b_0 = 0, \qquad w_0 = T_0 = \text{const.} = 2m_0,$$

the constant m_0 being thus determined.

From [114] we have

$$\theta_0 = e^{2\phi}\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right), \qquad \lambda_0 = \left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right).$$

As it is desired that C_1 shall be isothermic, it follows from (121) that we must have

$$\frac{c_1}{m_1} = 2 + f_1.$$

If we put

$$A = F, \qquad B = G + \frac{g^2}{4} + Fg + 4C,$$

equation (129) becomes

$$\Delta_1 \theta_0 + \left(\frac{\lambda_0 \theta_0}{2}\right)^2 + A\lambda_0 \theta_0 - 2D\lambda_0 - 2E\theta_0 + B = 0, \tag{169}$$

* The results which we obtain thus briefly are due to Darboux, Annales de l'École Normale Supérieure, ser. 3, vol. 16 (1899), and Bianchi, Annali di matematica, ser. 3, vol. 11 (1905), pp. 93–158.

and equation (131) reduces to

$$m_1(2m_1 + A)^2 - Bm_1 - 2DE = 0. \tag{170}$$

From (5) it follows that if θ_0 and λ_0 are of the form (136) for an isothermic surface, then ρ is a function of v alone and $\bar{S}$ is a surface of revolution, which we have excluded from the discussion. Accordingly an isothermic surface other than a surface of revolution which satisfies the condition (169) is called a *special isothermic surface of class* (A, B, D, E).

From (139) it follows that the transformation functions leading to a complementary transform of $\bar{S}$ are

$$\eta_1 = e^{\phi}\frac{\partial\lambda_0}{\partial u}, \qquad \beta_1 = -e^{\phi}\frac{\partial\lambda_0}{\partial v}, \qquad \mu_1 = w_1 = \tfrac{1}{2}\lambda_0\theta_0 + A + 2m_1,$$

$$\tau_1 = -\theta_1 = -\theta_0 - \frac{D}{m_1}, \qquad \sigma_1 = -\lambda_1 = -\lambda_0 - \frac{E}{m_1}.$$

In order that this value of w_1 shall be equal to that given by (128), we must have $g = 0$.

The results of §§ 11 and 12 when applied to this case enable us to state

THEOREM 14. *A special isothermic surface, whose constants are such that the roots of equation* (170) *are real and distinct, admits three complementary surfaces which are isothermic surfaces of the same class; and the circle-planes of the three transformations coincide.*

From § 13 we have

THEOREM 15. *When $\bar{S}$ is a special isothermic surface, the equations of a transformation D_m of this surface into an isothermic surface admit the first integral*

$$e^{\phi}\left(\eta_1\frac{\partial\lambda_0}{\partial u} - \beta_1\frac{\partial\lambda_0}{\partial v}\right) - (m\lambda_0 + E)\theta_1 - (m\theta_0 + D)\lambda_1 + \mu_1(A + 2m + \tfrac{1}{2}\theta_0\lambda_0) = \text{const.}$$

When this constant is zero, the resulting surface is a special isothermic surface of the same class as $\bar{S}$.

From equation (163) for the t's, it follows that if $\bar{S}$, $\bar{S}_1$, $\bar{S}_2$ are isothermic surfaces so also is $\bar{S}_{12}$. Hence

THEOREM 16. *If $\bar{S}_1$ and $\bar{S}_2$ are two special isothermic transforms of $\bar{S}$, there exists a fourth special isothermic surface of the same class which is a transform of $\bar{S}_1$ and $\bar{S}_2$.*

For the present case Theorem 13 becomes

THEOREM 17. *The parallel transform of a special isothermic surface of class (A, B, D, E) is a special surface of class (A, B, E, D).*

17. When θ_0 is linear in t

We consider throughout the remainder of this memoir the case where θ_0 is linear in t with constant coefficients. It follows at once from (5) and (104) that λ_0 is linear in r. Thus we put

$$\theta_0 = c_0 t + d_0, \qquad \lambda_0 = c_0 r + e_0. \tag{171}$$

From (4) we have by integration

$$T_0 = \frac{m_0}{c_0}(\theta_0 \lambda_0 + g_0), \tag{172}$$

where g_0 is a constant of integration.

In consequence of (10) we have from (116) and (128)

$$\begin{gathered} a_1 = \frac{a_0}{T_0}(c_0 - R_1), \qquad b_1 = \frac{b_0}{T_0}(c_0 - R_1), \\ w_1 = \frac{w_0}{T_0}(c_0 - R_1) + \frac{m_0(g - g_0)}{w_0}. \end{gathered} \tag{173}$$

It follows from (113) that if g and g_0 in (173) were equal the surfaces C_0 and C_1 would coincide, which evidently is contrary to hypothesis. If we put

$$h = m_0(g - g_0), \tag{174}$$

it follows from the preceding considerations that h does not vary in passing to the various transforms of C.

In considering equations (132) we excluded the case where θ_0 is of the form (171). Referring to (130) and (172), we see that the first two expressions in (132) vanish only when g and g_0 are equal, which we have seen to be impossible. Hence when θ_0 is of the form (171), the vanishing of the second terms of (132) are a consequence of (129) which reduces to

$$\begin{aligned} h\left\{\rho\left[\left(\frac{\partial t}{\partial u}\right)^2 + \left(\frac{\partial t}{\partial v}\right)^2\right] + 1\right\} \\ + 2T_0(Crt + Et + Dr + F + c_0) + jT_0^2 = 0, \end{aligned} \tag{175}$$

where j is the constant given by

$$hj = 2\frac{C}{c_0}(d_0 e_0 + g_0) - 2De_0 - 2Ed_0 + 2Fc_0 + G + c_0^2. \tag{176}$$

If we put

$$\begin{gathered} 2C + jm_0 c_0 = -hm_0' c_0', \qquad 2D + jm_0 d_0 = -hm_0' d_0', \\ 2E + jm_0 e_0 = -hm_0' e_0', \\ 2(F + c_0) + j\frac{m_0}{c_0}(e_0 d_0 + g_0) = -h\frac{m_0'}{c_0'}(e_0' d_0' + g_0'), \end{gathered} \tag{177}$$

equation (175) is reducible to

$$(178)\quad \rho\left[\left(\frac{\partial t}{\partial u}\right)^2+\left(\frac{\partial t}{\partial v}\right)^2\right]+1=T_0\frac{m_0'}{c_0'}[(c_0't+d_0')(c_0'r+e_0')+g_0'].$$

Suppose now that we have a surface C satisfying the equations of condition in § 1, with θ_0 and λ_0 of the form (171), and also equation (178). By means of (176) and (177) we can transform the equation (178) to the form (175). Thus from the five arbitrary constants m_0', c_0', d_0', e_0', g_0' we pass to seven, namely C, D, E, F, G, L, and j. Evidently then in all generality h and j can be given fixed values, so long as we take $h \neq 0$. The resulting equations assume a simple form, if we take $j = 0$ and $h = -2$. Now equation (175) becomes

$$(179)\quad \rho\left[\left(\frac{\partial t}{\partial u}\right)^2+\left(\frac{\partial t}{\partial v}\right)^2\right]+1=T_0(Crt+Et+Dr+F+c_0),$$

and the constants G and g which appear in equation (131) are given by

$$(180)\quad \begin{aligned} G &= 2De_0+2Ed_0-2Fc_0-c_0^2+2\frac{C}{c_0}(d_0e_0+g_0),\\ g &= g_0-\frac{2}{m_0}. \end{aligned}$$

We recall that every surface C has in general two conjugate surfaces, C_0 and C_0'. From [56] we have that the functions a_0', b_0', w_0' determining C_0' are given by

$$(181)\qquad a_0'=\frac{a_0}{w_0^2},\qquad b_0'=\frac{b_0}{w_0^2},\qquad w_0'=-\frac{1}{w_0},\qquad T_0'=\frac{T_0}{w_0^2}.$$

Since the left-hand member of (179) is equal to $H^2\rho^2$, it follows from (10) that this equation may be written

$$(182)\qquad \frac{T_0}{w_0^2}=Crt+Et+Dr+F+c_0.$$

In consequence of (106), (181), and (182) equations (173) may be written

$$(183)\quad a_1=a_0'+\frac{m_1}{m_0}a_0,\qquad b_1=b_0'+\frac{m_1}{m_0}b_0,\qquad w_1=w_0'+\frac{m_1}{m_0}w_0.$$

Expressing the condition that these expressions shall satisfy equations (113), when a_0, b_0, w_0 and a_0', b_0', w_0' satisfy it, we get

$$(184)\quad \begin{aligned} m_1\theta_1 &= m_0\theta_0+m_0'\theta_0',\\ m_1\lambda_1 &= m_0\lambda_0+m_0'\lambda_0'. \end{aligned}$$

Equations (183) show that each circle-plane of the complementary trans-

formations of C is the plane of the corresponding points M, M_0, M_0' on C, C_0, C_0' respectively, and M_1 lies on the line $M_0 M_0'$.

From equations (184), (171), (99), and (103) it follows that θ_0' and λ_0' are linear functions of t and r respectively.

Conversely, we shall show that when θ_0 and θ_0' are linear functions of t, the surface C is a special surface. If we take

$$\theta_0' = c_0' t + d_0', \qquad \lambda_0' = c_0' r + e_0',$$

the function T_0' is given by an equation analogous to (172), namely

$$T_0' = \frac{m_0'}{c_0}(\theta_0' \lambda_0' + g_0'). \tag{185}$$

From (181) and (10) we have

$$H^2 \rho^2 = T_0 T_0', \tag{186}$$

which is the same as (178) in consequence of (185). Hence we have

THEOREM 18. *When θ_0 and θ_0' for a surface C are linear in t with constant coefficients, C admits three complementary transforms, in general distinct, whose circle-planes are the planes determined by corresponding points on C, C_0, and C_0'; moreover, this is the only type of special surface C with θ_0 linear in t.*

We propose to show that the functions θ_{10} and λ_{10} of a complementary transform are linear in t_1 and r_1 respectively.

We put

$$\theta_{10} = (c_0 + c')t_1 + (d_0 + d'), \qquad \lambda_{10} = (c_0 + c')r_1 + (e_0 + e'). \tag{187}$$

If these equations be multiplied by $m_1 \lambda_1$ and $m_1 \theta_1$ respectively and the expressions for $m_1 \lambda_1 \theta_{10}$, $m_1 \theta_1 t_1$, $m_1 \lambda_1 t_1$, and $m_1 \theta_1 r_1$ from (140), (121), and (146) be substituted, the resulting equations vanish identically if

$$(D + m_1 d_0)c' = (C + m_1 c_0)d',$$

$$\left[\frac{m_1}{c_0}(e_0 d_0 + g_0) + F - c_1\right]c' = (E + m_1 e_0)d' + \left[\tfrac{1}{2}(G + c_0^2) + c_0 c_1 - 2m_1\left(g_0 - \frac{1}{m_0}\right)\right],$$

$$(E + m_1 e_0)c' = (C + m_1 c_0)e',$$

$$\left[\frac{m_1}{c_0}(e_0 d_0 + g_0) + F - c_1\right]c' = (D + m_1 d_0)e' + \left[\tfrac{1}{2}(G + c_0^2) + c_0 c_1 - 2m_1\left(g_0 - \frac{1}{m_0}\right)\right].$$

As these four equations in c', d', e' are consistent, we can find their values such that (187) shall be satisfied. Hence we have

THEOREM 19. *When θ_0 for a special surface C is a linear function of t, the function θ_{10} of each of the complementary transforms of C is linear in t_1.*

From (130), (172), and (180) we have

$$N = c_0 - \frac{2}{T_0}. \tag{188}$$

Hence the transformation functions of complementary transforms, namely (139), are reducible now to

$$\begin{gathered}\mu_1 = c_1 + c_0 - \frac{2}{T_0}, \qquad \tau_1 = -\frac{2t}{T_0} - \frac{d_0 m_1 + D}{m_1}, \\ \eta_1 = -\frac{2\sqrt{\rho}}{T_0}\frac{\partial t}{\partial u}, \qquad \beta_1 = -\frac{2\sqrt{\rho}}{T_0}\frac{\partial t}{\partial v}, \\ m_1 \tau_1 \sigma_1 = \mu_1 (R_1 + c_1) + m_1 \lambda_1 \theta_1.\end{gathered} \tag{189}$$

18. SPECIAL SURFACES OF GUICHARD

In this section we apply the results of the preceding section to establish the existence of special surfaces of Guichard and to show that they admit transformations.*

When we compare equations (58) and (64), we note that for every surface of Guichard θ_0 and λ_0 are of the form (171) where

$$c_0 = -\frac{1}{2m_0}, \qquad d_0 = e_0 = 0.$$

Again comparing (63) and (172), we see that

$$g_0 = -\frac{1}{4m_0^2}.$$

In consequence of (58), (61), and (63), equations (189) become

$$\begin{gathered}\mu_1 = 2(h - \overline{m}_1), \qquad \tau_1 = -2(e^t + \tfrac{1}{2}d_1), \\ \eta_1 = -2\operatorname{csch}\alpha\frac{\partial \xi}{\partial u}, \qquad \beta_1 = -2\operatorname{sech}\alpha\frac{\partial \xi}{\partial v}, \\ m_1 \tau_1 \sigma_1 = 2D(h^2 - 1)e^{-t} + 2e^t E + 2h(c_1 - F - C) \\ + \left[(\overline{m}_1 + 1)\frac{m_1}{m_0} + 2\overline{m}_1(F - c_1) + De_1 - 2C\right],\end{gathered} \tag{190}$$

* The results here obtained were established at length by the author in a previous memoir, cf. Annali di matematica, ser. 3, vol. 22 (1914), pp. 191–248.

where $\overline{m}_1$ is the constant given by

$$\overline{m}_1 = 1 - \tfrac{1}{2}\left(\frac{C}{m_1} - \frac{1}{2m_0}\right) = 1 - \tfrac{1}{2}(c_1 + c_0). \tag{191}$$

If the last of (190) be differentiated with respect to u, the resulting equation is reducible by means of (81) to

$$m_1 \tau_1 e^{-\xi}(k_1 \phi_1 + \psi_1) + m_1 \sigma_1 e^{\xi} \sinh \alpha + Ee^{\xi} \sinh \alpha$$
$$+ e^{-\xi} D[2h(\cosh \alpha + h \sinh \alpha) + (1 - h^2) \sinh \alpha]$$
$$+ (c_1 - F - C)(\cosh \alpha + h \sinh \alpha) = 0,$$

where now k_1, as defined by (82), is given by

$$k_1 = \frac{hd_1 + 2e^{\xi}\overline{m}_1}{d_1 + 2e^{\xi_1}}.$$

The consistency of this equation with equations (190) necessitates the relation

$$c_1 - F - C = 4m_1 \overline{m}_1, \tag{192}$$

under which condition the last of (190) reduces to

$$m_1 \tau_1 \sigma_1 = 2D(h^2 - 1)e^{-\xi} + 2e^{\xi} E + 8hm_1 \overline{m}_1$$
$$+ (De_1 - (1 + \overline{m}_1^2)4m_1). \tag{193}$$

When these values are substituted in (27), we obtain

$$\operatorname{csch}^2 \alpha \left(\frac{\partial \xi}{\partial u}\right)^2 + \operatorname{sech}^2 \alpha \left(\frac{\partial \xi}{\partial v}\right)^2 \cdot$$
$$+ h^2 + D(1 - h^2)e^{-\xi} - Ee^{\xi} + 2Bh + H = 0, \tag{194}$$

where

$$B = -(2m_1 + 1)\overline{m}_1, \qquad H = 2m_1 + \overline{m}_1^2(1 + 2m_1) - \tfrac{1}{2} De_1. \tag{195}$$

On the other hand equation (179) reduces to (194) where now

$$H = 1 - C - F - c_0 = 2C - 1 + 2\overline{m}_1(2m_1 + 1). \tag{196}$$

Equating these two expressions for H, we have

$$2C + \tfrac{1}{2} De_1 = (1 + 2m_1)(\overline{m}_1 - 1)^2. \tag{197}$$

For the present case we have from (180)

$$G = 2Cc_0 - 2Fc_0 - c_0^2, \qquad g = -c_0^2 + 4c_0,$$

and the cubic (131) becomes

$$2Fc_0 + c_0^2 - 2Cc_0 + 2m_1(4c_0 - c_0^2) + \frac{2CF}{m_1} - \frac{2DE}{m_1} - \frac{C^2}{m_1^2} = 0.$$

In consequence of (191) and (192) this equation is equivalent to (197). With the aid of (195) it can be shown to be equivalent also to

$$(198)\qquad (H-2m_1)2m_1(2m_1+1)-B^2\,2m_1+DE(2m_1+1)=0,$$

which is the equation found in the above-mentioned memoir.

When the above values of the constants are substituted in the expression

$$\tfrac{1}{2}(G+c_0^2)+c_0c_1-2m_1\left(g_0-\frac{1}{m_0}\right),$$

it vanishes identically. Hence the quantities c', d', e' in (187) are equal to zero, and we have

$$(199)\qquad \theta_{10}=c_0t_1+d_0,\qquad \lambda_{10}=c_0r_1+e_0.$$

As the functions μ_1, η_1, β_1, τ_1, σ_1, given by (190) and (193), satisfy equations (81), we have

THEOREM 20. *When a surface of Guichard satisfies the condition* (194), *in which the four constants* B, D, E, *and* H *are such that the roots of equation* (198) *are real and distinct, the surface admits three complementary transforms which also satisfy* (194).

The latter part of this theorem follows from the fact that the transforms are surfaces of Guichard, and from the general results of § 12, since H, as given by (196), depends only on C and F, and B, as follows from (195) and (196), is equal to

$$(200)\qquad B=\tfrac{1}{2}(3C+F+c_0-2).$$

When a surface of Guichard satisfies (194) we say that it is a *special surface of class* (B, D, E, F).

Since $c_0=-\frac{1}{2}m_0$ and $d_0=0$, it follows from (64) that

$$(201)\qquad r=-(1+h)e^{-\xi}.$$

When $\bar{S}$ is a special surface of Guichard and $\bar{S}_2$ is a surface of Guichard which is a transform of $\bar{S}$ in accordance with §§ 5 and 6, then in consequence of the formulas in these sections and the present one, equation (158) is reducible to

$$(202)\qquad \begin{aligned}&\operatorname{csch}\alpha\frac{\partial\xi}{\partial u}\eta_2+\operatorname{sech}\alpha\frac{\partial\xi}{\partial v}\beta_2-\left(\frac{D}{2}+m_2e^{\xi}\right)\left(\sigma_2+\frac{\mu_2^2}{\tau_2}\right)\\&\quad+\tau_2\left[m_2(1-h^2)e^{-\xi}+\frac{E}{2}\right]-\mu_2[B+h(1+2m_2)]=0.\end{aligned}$$

Hence for all transformations from a special surface of Guichard into a surface of Guichard the left-hand member of this equation is constant. Applying Theorem 10 to this case, we have

THEOREM 21. *When a special surface of Guichard undergoes a transformation A_m for which the functions satisfy* (202), *the resulting surface is a special surface of Guichard of the same class.*

If $\overline{S}$, $\overline{S}_1$, and $\overline{S}_2$ are three surfaces of Guichard of which $\overline{S}_1$ and $\overline{S}_2$ are transforms of $\overline{S}$, the fourth surface of the quatern, determined in accordance with the general theorem of permutability of transformations A_m is a surface of Guichard. We shall not go through the details of establishing this here.* Hence in view of Theorem 12 we have

THEOREM 22. *If $\overline{S}_1$ and $\overline{S}_2$ are two transforms of a special surface of Guichard $\overline{S}$ satisfying* (202), *the fourth surface of the quatern is a special surface of the same class.*

Combining Theorems 3 and 13, we have

THEOREM 23. *The parallel transform of a special surface of Guichard of class $(B, D; E, H)$ is a special surface of class (B, E, D, H).*

19. ENVELOPE OF THE CIRCLE-PLANES OF A COMPLEMENTARY TRANSFORMATION

We consider now the envelope of the circle-planes of a complementary transformation of a special surface.

With the aid of (34), (41), and (139) the expressions for the functions p, ω, and ψ, as given by (47), (50′), and (51), are reducible to

$$p = \frac{\theta_0}{\nu}, \qquad \omega = \left[\theta_0(Dr + F - c_1) - (Ct + D)\left(\frac{T_0}{m_0} - \lambda_0 t\right)\right]\frac{1}{\nu},$$

$$\psi = \frac{\theta_1}{\nu}\left[C\left(\lambda_0 t - \frac{T_0}{m_0}\right) + \lambda_0 D - m_1 g\right], \tag{203}$$

$$\nu = \theta_0(Cr + E) - \lambda_0(Ct + D).$$

We recall that the linear element of the envelope S_0 is

$$ds_0^2 = d\omega^2 + 2dpd\psi.$$

Hence, if we put

$$\omega = x, \qquad p = -y + iz, \qquad 2\psi = -(y + iz), \tag{204}$$

the surface whose coördinates x, y, z, are given by (204) is applicable to S_0.

When in particular $\overline{S}$ is a special isothermic surface, the expressions (203) are reducible in consequence of the results to § 16, to

$$p = \frac{\theta_0}{\nu}, \qquad \omega = -\frac{2}{\nu}(m_1\theta_0 + D),$$

$$\psi = \frac{m_1\theta_0 + D}{\nu}(\lambda_0 d_1 - 2c_1), \qquad \nu = \theta_0 E - \lambda_0 D.$$

* Cf. Annali, l. c., p. 213.

Substituting these values in (204) and eliminating θ_0 and λ_0, we get

$$(y + iz)[x - 2m_1(y - iz)] = 2x[(2C + d_1 E)(y - iz) - c_1 x + d_1].$$

Hence S_0 is applicable to a quadric, as Darboux has shown.*

When $\overline{S}$ is a special surface of Guichard, in consequence of the results of § 18, the expressions (203) are reducible to

$$p = \frac{e^{\xi}}{2\nu}, \qquad \omega = -\frac{2m_1\overline{m}_1 e^{\xi} + Dh}{\nu},$$

$$\psi = -\left[m_1(1 - m_1)e^{\xi} + \frac{D}{2}(1 - h)\right]\frac{d_1 e^{-\xi}(1 + h) + 2(1 + \overline{m}_1)}{\nu},$$

$$\nu = \tfrac{1}{2} Ee^{\xi} + \tfrac{1}{2} De^{-\xi}(1 - h^2).$$

These are equivalent to the expressions for p, ω and ψ which we have found previously, and from which we showed that S_0 is applicable to a quadric.†

20. The Case $m_1 = m_2$

In this closing section we consider the case where two transformations A_m, with the same values of the constant m, have the same circle-planes. From (93) and (98) it follows that in this case θ_0 and λ_0 are of the form (171).

If we put $m_1 = m_2 = m$, equations (98) may be replaced by

$$\theta_2 = \theta_1 + \frac{\kappa m_0}{m}\theta_0, \qquad \lambda_2 = \lambda_1 + \frac{\kappa m_0}{m}\lambda_0. \tag{205}$$

From these equations, (95) and (10) we have the first two of the equations

$$a_2 = a_1 + \kappa a_0, \qquad b_2 = b_1 + \kappa b_0, \qquad w_2 = w_1 + \kappa w_0, \tag{206}$$

the last being a consequence of the fact that a_2 and b_2 as thus given must satisfy equations (113) for $i = 2$, when a_1 and b_1 satisfy these for $i = 1$.

When we express the condition that θ_2 and θ_1 shall satisfy equations of the form (36), the resulting equations are reducible in consequence of (10) and (171) to the single one

$$m\lambda_2 t_2 = m\lambda_1 t_1 - \kappa m_0 d_0 r + \kappa B_0, \tag{207}$$

where

$$B_0 = \frac{c_0}{m} - \frac{g_0}{c_0} - \frac{e_0 d_0}{c_0}.$$

It is readily shown that this value of t_2 satisfies equations (38) for $i = 2$ on the assumption that t_1 satisfies them for $i = 1$.

* L. c.; cf. also Bianchi, l. c., p. 139.

† Annali, l. c., p. 241.

The remaining condition to be satisfied in order that the transform of C by $a_2, b_2, \omega_2, \theta_2, \lambda_2$ shall be a surface C is equation (115) for $i = 2$. This necessitates the relation

$$m_0(\lambda_0\theta_1 + \theta_0\lambda_1) - \Phi_{10} - \left(T_0 - \frac{m_0 c_0}{m}\right) m\lambda_1(t_1 - t) = \kappa\frac{m_0^2}{m}\left(g_0 - \frac{c_0^2}{2m}\right). \tag{208}$$

If the left-hand member of this equation be differentiated with respect to u and v separately, the resulting equations vanish identically in consequence of (36) and (96). Hence for any transformation K_m of a surface C satisfying (171) the left-hand member of (208) is a constant. If this constant is different from zero, a constant κ can be found so that (208) holds, and then (205) and (206) define a second transformation such that for the two transformations the circle-planes coincide. Accordingly we have

THEOREM 24. *When the function θ_0 of a surface S is linear in t, each transformation A_m for whose functions the left-hand member of* (208) *is different from zero has associated with it another transformation A_m such that the circle-planes of the two transformations coincide.*

The surfaces of Guichard are of this type.

PRINCETON UNIVERSITY,
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ON FIGURES OF EQUILIBRIUM OF A ROTATING COMPRESSIBLE FLUID MASS; CERTAIN NEGATIVE RESULTS*

BY

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1. **Introduction.** The problem of determining from theoretical considerations the shape of celestial bodies has led to extensive discussions of figures of equilibrium of a rotating fluid mass. Especial difficulties have been encountered when the fluid is assumed compressible, the case in nature. They are in fact so great that various writers have considered it worth while to establish that certain figures, for a certain type of law of density of the fluid mass, are *not* figures of equilibrium. Thus Hamy† proved that an ellipsoidal mass composed of a finite number of layers of homogeneous fluid in which the density increases layer by layer from surface to center and the bounding surfaces of the layers are ellipsoids coaxial with the surface ellipsoid, cannot be a figure of equilibrium. Véronnet‡ extended Hamy's theorem to the case in which the density is any continuous function increasing from surface to center. Volterra§ arrived at the same negative conclusion, supposing the equidensity surfaces to be homothetic ellipsoids, and the density function to be any positive, finite, integrable function not a constant.

In this paper other negative theorems are established. Surfaces of great generality are considered, as contrasted with the ellipsoidal surfaces discussed by earlier writers, but generality in the conclusion is lost in so far as there may be for a given surface a finite number of values of a certain compressibility parameter for which the conclusion is not true. The surfaces discussed include all ellipsoids, and the results for ellipsoids are new since it is not assumed that equidensity surfaces are ellipsoids.

2. **Notation and fundamental equations.** A fluid mass is considered whose surface is Σ and density at any point P is ρ; such a mass is referred to briefly

* Presented to the Society, under a slightly different title, December 27, 1913.

† Thesis (Paris), *Étude de la figure des corps célestes*, 1887. Reproduced by Tisserand, *Traité de Mécanique Céleste*, vol. 2, pp. 186–192.

‡ Thesis (Paris), *Rotation de l'ellipsoide hétérogène et figure exacte de la Terre*, 1912, Journal de Mathématiques, ser. 6, vol. 8 (1912), pp. 331–465.

§ *Sur la stratification d'une masse fluide en équilibre*, Acta Mathematica, 27 (1903), pp. 105–124.

as *the mass* (Σ, ρ). The mass is rotating with an angular velocity ω about an axis A; the angular velocity is constant throughout the mass. The only forces acting arise from the mutual attractions of the particles under the newtonian law, and from the internal pressure.

In order for the fluid mass to be in equilibrium the following conditions must be satisfied:

I. At every point of the fluid

$$dp = f\rho dU, \tag{1}$$

where dp and dU are the differentials of the pressure and of the potential of the attraction and centrifugal force, and where f is the gravitation constant.

II. On the surface Σ the potential U and the density ρ are constant.

III. The center of gravity of the mass (Σ, ρ) lies on the axis A of rotation.

If, for a mass (Σ, ρ) rotating with an angular velocity ω about an axis A, conditions I, II, III are satisfied, then $(\Sigma, \rho, \omega, A)$ will be called a *system of equilibrium.*

Turning to the equations of the problem, we have first

$$U = \frac{\omega^2 R^2}{2f} + \int \frac{\rho d\tau}{\Delta}, \tag{2}$$

where R is the distance from the axis of rotation A to the point P at which U is computed, and where Δ is the distance from P to the volume element $d\tau$, and the integral is extended throughout the mass (Σ, ρ). Next we assume a relation between the pressure and density in the mass to hold, expressed by the equation

$$hdp = f\rho \frac{\partial F(\rho, h)}{\partial \rho} d\rho, \tag{3}$$

where h is a compressibility parameter, and $F(\rho, h)$ is a polynomial* in ρ, is analytic in h in the vicinity of $h = 0$, and is such that

$$\frac{\partial F(\rho, h)}{\partial \rho}\Big]_{\rho=\rho_0} \neq 0,$$

if $\rho_0 > 0$, ρ_0 being a constant.

Equations (1) and (3) are satisfied if and only if

$$\frac{\partial F(\rho, h)}{\partial \rho} d\rho = hdU \qquad \text{or} \qquad \rho = 0.$$

The latter equation may be discarded in our problem. The solutions of the former under the condition that $\rho = \rho_0 > 0$ at a fixed point P_0 in the mass (Σ, ρ) are considered; they are solutions of

$$F(\rho_1, h) = F(\rho_0, h) + h(U_1 - U_0), \tag{4}$$

* The function $F(\rho, h)$ may be greatly generalized in the later argument; see § 7.

where subscripts are used to indicate the point at which the functional values are taken, P_1 being a variable point in the mass (Σ, ρ). In view of equation (2), we may write (4) in the form

$$F(\rho_1, h) = F(\rho_0, h) + \frac{h\omega^2}{2f}(R_1^2 - R_0^2) + h\int \kappa_{12}\,\rho_2\,d\tau_2, \tag{5}$$

$$\kappa_{12} = \frac{1}{\Delta_{12}} - \frac{1}{\Delta_{02}},$$

where Δ_{12} is the distance from P_1 to the volume element $d\tau_2$, and Δ_{02} has a similar meaning.

Equation (5) is the *fundamental integral equation* which the density function must satisfy. It has its simplest form when $F(\rho, h) = \rho$. In this case (3) is the Laplace pressure-density equation*

$$hdp = f\rho d\rho; \tag{3'}$$

and (5) may be written

$$\rho_1' = 1 + hq(R_1^2 - R_0^2) + h\int \kappa_{12}\,\rho_2'\,d\tau_2, \tag{5'}$$

where

$$\rho_1' = \rho_1/\rho_0, \qquad q = \omega^2/2f\rho_0. \tag{5'a}$$

3. **More exact statement of the problem.** The general problem of determining figures of equilibrium from our point of view may now be stated thus: Given an axis of rotation A and a pressure-density equation of the form (3), it is required to find an angular velocity ω, a surface Σ, and a constant $\rho_0 > 0$, such that a solution ρ of equation (5) makes $(\Sigma, \rho, \omega, A)$ a system of equilibrium. Our method will be to make the compressibility parameter h play a fundamental rôle; that is, various functions will be expanded as power series in this parameter. In this paper† we shall let Σ be any one of an extensive class of surfaces and P_0 any interior point, and consider the possibility of determining ω and ρ_0 as functions of h such that when equation (5) is satisfied $(\Sigma, \rho, \omega, A)$ is a system of equilibrium for some continuous range of values of h. The negative conclusion we shall reach shows that to solve the problem by our method it is necessary to let Σ vary with h.

In order to describe Σ accurately, we make the following definitions. By a *regular surface* we understand one which can be cut into a finite number of pieces, each of which has at every point a continuously turning tangent plane; by a *simple surface*, one which is continuous and has no multiple points or lines. *The surface Σ will be chosen from the class of closed, simple, regular surfaces.*

We shall assume that ω and ρ_0 can be determined as desired and derive

* The pressure-density equation $dp = h\rho d\rho$ was used by Laplace (*Mécanique Céleste*, book XI, chap. IV); it is to be observed that our h is proportional to the reciprocal of the Laplace h. For us, $h = 0$ gives an incompressible fluid.

† In a later paper I propose to adopt a different method, and let Σ vary with h.

necessary consequences. We first consider the special case when (5) reduces to the simpler equation (5′).

4. **The solution of the integral equation** (5′). This is a linear integral equation of the Fredholm type, but presents an unusual difficulty in the fact that the kernel κ_{12} becomes infinite not only when the variable points P_1 and P_2 coincide, but also when P_2 coincides with the fixed point P_0. This difficulty is met by considering the iterated equation*

$$\rho_1 = \eta_1 + h^2 \int \kappa_{12}^{(2)} \rho_2 \, d\tau_2 , \tag{7}$$

where we have dropped accents and written

$$\eta_1 = 1 + hq(R_1^2 - R_0^2) + h \int \kappa_{12} [1 + hq(R_2^2 - R_0^2)] \, d\tau_2 ,$$

$$\kappa_{12}^{(2)} = \int \kappa_{13} \kappa_{32} \, d\tau_3 \tag{8}$$

$$= \int \left[\frac{1}{\Delta_{13} \Delta_{32}} - \frac{1}{\Delta_{03} \Delta_{32}} \right] d\tau_3 - \frac{1}{\Delta_{02}} \int \left[\frac{1}{\Delta_{13}} - \frac{1}{\Delta_{03}} \right] d\tau_3 .$$

The function η_1 is readily seen to be continuous in P_1 throughout Σ; moreover the last two integrals are continuous functions† in P_1 and P_2 throughout Σ, and hence we may write‡

$$\kappa_{12}^{(2)} = \mu_{12} + \mu'_{12} \cdot \frac{1}{\Delta_{02}} ,$$

where the only discontinuity comes in through the factor $1/\Delta_{02}$. If we use polar coördinates with the pole at P_0, we have

$$\Delta_{02} = r_2 , \qquad d\tau_2 = r_2^2 \sin \theta_2 \, dr_2 \, d\theta_2 \, d\phi_2 ,$$

and hence the function η_1 and the kernel $\kappa_{12}^{(2)} r_2^2 \sin \theta_2$ are continuous in (r_1, θ_1, ϕ_1) and $(r_1, \theta_1, \phi_1, r_2, \theta_2, \phi_2)$ respectively. Thus the apparent difficulty in equation (7), due to the way in which the kernel becomes infinite, disappears.

We know then from the general theory of integral equations,§ that equa-

* That is, the equation obtained from (5′) by changing the subscripts 1 and 2 to 2 and 3 and substituting the right member for ρ_2 in the integral of (5′), and then altering the notation slightly. It is here, where an interchange of order of integration occurs, and later in this article in connection with solving (7), that some assumption concerning the character of Σ is chiefly needed. It could be of less restricted class.

† The continuity of the function given by the first integral follows from the inequalities given by Fredholm, Acta Mathematica, vol. 27 (1903), p. 387. The second integral is the potential for a constant density, which is known to be continuous.

‡ As a matter of fact μ'_{12} is independent of P_2.

§ See, for example, Bôcher, *An Introduction to the Study of Integral Equations;* or Lalesco, *Introduction à la Théorie des Équations Intégrales.*

tion (7) has, except for isolated finite values of h, a unique solution continuous in P_1, and it readily follows that the same is true of (5'). Also we may write the solution

$$\rho_1 = \eta_1 + h^2 \int \mathrm{K}_{12}^{(2)} \eta_2 \, d\tau_2, \tag{9}$$

where $\mathrm{K}_{12}^{(2)}$ is, except for isolated finite values of h, analytic in h and continuous in P_1 and P_2 throughout Σ. In particular, $\mathrm{K}_{12}^{(2)}$ is analytic in h at $h = 0$.

By use of equation (8) we write

$$\rho_1 = 1 + hq(R_1^2 - R_0^2) + hS_1(1) + h^2 qS_1(R^2 - R_0^2), \tag{10}$$

where, if f is any function whose value at P_i is f_i,

$$S_1(f) = \int \kappa_{12} f_2 \, d\tau_2 - h \int \mathrm{K}_{12}^{(2)} \Big[f_2 + h \int \kappa_{23} f_3 \, d\tau_3 \Big] d\tau_2 .$$

We observe that when f is independent of h, $S_1(f)$ is analytic in h except for isolated finite values of h, and in particular is analytic at $h = 0$.

5. **The first conclusion.** If ω and ρ_0 can be determined so that when ρ_1 is a solution of (5') $(\Sigma, \rho, \omega, A)$ is a system of equilibrium for other than isolated finite values of h, then it is possible to determine $q\,(= \omega^2/2f\rho_0)$ so that the function ρ_1 given by (10) is effective in satisfying equilibrium conditions II and III. In the first place q may be so chosen that the values of ρ at any two points P_1 and P_2 on Σ are the same. The value of q for this purpose is readily found to be

$$q = \frac{S_2(1) - S_1(1)}{R_1^2 - R_2^2 + h\,[\,S_1(R_1^2 - R_0^2) + S_2(R_1^2 - R_0^2)\,]}, \tag{11}$$

where subscripts on the S's indicate the points at which they are computed. Since $q\,(= \omega^2/2f\rho_0)$ is constant as far as P_1 and P_2 are concerned, its characterization as to h is obtained by choosing P_1 and P_2 so that $R_1 \neq R_2$. We thus find that q as given by (11) is analytic in h except for isolated finite values, and in particular is analytic at $h = 0$. This value of q, therefore, when substituted in (10), determines a function ρ which is in general analytic in h, in particular in the vicinity of $h = 0$, and which for all such values of h is constant on Σ. Also since the center of gravity of (Σ, ρ) is on the axis A for other than isolated finite values of h, it is in general on A, and in particular for all values of h near $h = 0$.

The argument thus far reduces the consideration of values of h to those in the vicinity of $h = 0$. We now write

$$\rho_1 = \sum_{i=0}^{\infty} \rho_1^{(i)} h^i; \qquad q = \sum_{i=0}^{\infty} q^{(i)} h^i, \tag{12}$$

where $\rho_1^{(i)}$ is continuous throughout Σ. Equilibrium conditions I and II *require* (*a*) *that* $\rho_1^{(i)}$ *be constant on* Σ, and (*b*) *that the center of gravity of* $(\Sigma, \rho^{(i)})$ *lie on the axis* A.

By substituting (12) into (5′) and equating coefficients of like power of h, we find that

$$\begin{aligned} &\rho^{(0)} = 1, \\ &\rho^{(i)} = q^{(i-1)}(R_1^2 - R_0^2) + \int \kappa_{12}\, \rho_2^{(i-1)}\, d\tau_2 \qquad (i = 1, 2, 3, \cdots). \end{aligned} \tag{13}$$

The right member of (13) is, except for an added constant, the potential at P_1, for a certain angular velocity, of the mass $(\Sigma, \rho^{(i-1)})$. In view of the italicized statement in the preceding paragraph, taking $i = 1$, we now have the following negative result.

THEOREM 1. *Let an axis* A *of rotation and a closed, simple, regular surface** Σ *be chosen arbitrarily. Then,* (*a*) *if* Σ *is not an equipotential surface of the mass* $(\Sigma, 1)$ *when rotating with a suitable constant angular velocity, or* (*b*) *if the center of gravity of the homogeneous mass* $(\Sigma, 1)$ *is not on* A, *there exist only isolated finite values of* h *for which* ω *and* ρ_0 *may be so chosen that, if* ρ *is a continuous solution of* (5′), $(\Sigma, \rho, \omega, A)$ *is a system of equilibrium.*

6. **The case of an ellipsoid.** If Σ is an ellipsoid and A is a diameter, the center of gravity of the mass $(\Sigma, 1)$ lies on A, but Σ may or may not be an equipotential surface of $(\Sigma, 1)$ when rotating with a proper angular velocity; if it is not, Theorem 1 applies; if it is, we proceed one step further.

For this purpose, we use the following method of computing the newtonian potential. Let Σ be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{14}$$

and let us write

$$\mu = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}; \qquad \nu = 1 - \frac{x^2}{a^2 + s} - \frac{y^2}{b^2 + s} - \frac{z^2}{c^2 + s}.$$

Then if the density is constant on the surface $\mu = \text{const.}$, the newtonian potential is given by†

$$V = \int_0^\infty \int_0^\nu \rho(\mu)\, d\mu \frac{ds}{D(s)}, \qquad D(s) = \frac{\sqrt{(a^2 + s)(b^2 + s)(c^2 + s)}}{\pi abc}. \tag{15}$$

Now suppose P_0 is taken on the axis A, so that $R_0 = 0$. Taking $i = 1$

* A more extensive class of surfaces might be used; e. g., Σ might be composed of a finite number of non-intersecting closed, simple, regular surfaces, and the conclusion still be true.

† This formula was used by Volterra, loc. cit., p. 108, with a slightly different notation. The proof is readily made by comparison with pp. 138–140 of Mathieu's *Théorie du Potential*, where a similar formula is derived for the potential at an exterior point.

in (13), we have then from (15)

$$\rho_1^{(1)} = q^{(0)} R_1^2 - M_1 x_1^2 - M_2 y_1^2 - M_3 z_1^2, \tag{16}$$

where

$$M_1 = \int_0^\infty \frac{ds}{(a^2+s)D(s)}, \qquad M_2 = \int_0^\infty \frac{ds}{(b^2+s)D(s)},$$

$$M_3 = \int_0^\infty \frac{ds}{(c^2+s)D(s)}.$$

It is readily proved that since $\rho_1^{(1)}$ is constant on Σ_1, and hence the right member of (16) is constant in x_1 and y_1 when z_1 is eliminated by use of (14), the axis A of rotation must be a principal axis of Σ. Suppose A is the z-axis. On eliminating z_1 from (16) and equating the coefficients of x_1^2 and y_1^2 to zero, we have

$$q^{(0)} - M_1 - \frac{c^2}{a^2} M_3 = 0, \qquad q^{(0)} - M_2 - \frac{c^2}{b^2} M_3 = 0.$$

The ellipsoid must therefore have axes so related that for some value of $q^{(0)}$ these two equations hold simultaneously.* For such ellipsoids

$$\rho_1^{(1)} = -c^2 M_3 \left[\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2}\right] = c^2 M_3 (\mu - 1). \tag{17}$$

We consider next $\rho^{(2)}$. From (13) we have

$$\rho_1^{(2)} = q^{(1)} R_1^2 + \int \kappa_{12} \rho_2^{(1)} d\tau_2.$$

The integral is, except for an added constant, the newtonian potential for the density function $\rho_2^{(1)}$. It may therefore be computed by use of (15) and (17). It is seen that $\rho_1^{(2)}$ contains terms of second and fourth degrees in x_1, y_1, z_1; the latter are

$$\frac{c^2 M_3}{2} \int_0^\infty \left(\frac{x_1^2}{a^2+s} + \frac{y_1^2}{b^2+s} + \frac{z_1^2}{c^2+s}\right)^2 \frac{ds}{D(s)}.$$

When z_1 is eliminated by use of (14) the fourth degree terms must vanish; hence

$$\int_0^\infty \frac{(a^2-c^2)\,sds}{(a^2+s)(c^2+s)D(s)} = 0; \qquad \int_0^\infty \frac{(b^2-c^2)\,sds}{(b^2+s)(c^2+s)D(s)} = 0.$$

Hence we have $a = b = c$; that is Σ is a sphere. Hence

THEOREM 2. *Even if Σ is an ellipsoid, not a sphere, and A is a diameter of Σ, the conclusion of Theorem* 1 *is true.*

* These conditions are satisfied if the ellipsoid is of revolution or if it is one of the figures of equilibrium discovered by Jacobi—the Jacobian ellipsoids.

7. The more general pressure-density equations. In the preceding paragraphs our equations come from the initial use of the Laplace pressure-density equation (3′). We now consider the consequences of the initial use of the more general equation (3). We choose the surface Σ and axis A as in the preceding discussion and attempt the formal determination of ρ_0, ρ, and ω^2 as functions analytic in h in the vicinity of $h = 0$ such that equation (5′) and equilibrium conditions II and III are satisfied for all values of h in the vicinity of $h = 0$. If we write

$$\rho_0 = \sum_{i=0}^{\infty} \rho_0^{(i)} h^i; \qquad \rho = \sum_{i=0}^{\infty} \rho^{(i)} h^i; \qquad \frac{\omega^2}{2f} = \sum_{i=0}^{\infty} \omega^{(i)} h^i, \tag{18}$$

conditions II and III require that $\rho^{(i)}$ be constant on Σ and that the center of gravity of the mass $(\Sigma, \rho^{(i)})$ be on A.

If we substitute (18) in (5) and take the value at $h = 0$ of $F(\rho_1, h)$ and its first and second h-derivatives,* we have

$$F(\rho_1^{(0)}, 0) = F(\rho_0^{(0)}, 0), \tag{19a}$$

$$\frac{\partial F(\rho_1^{(0)}, 0)}{\partial \rho_1} \cdot \rho_1^{(1)} + \frac{\partial F(\rho_1^{(0)}, 0)}{\partial h} = \frac{\partial F(\rho_0^{(0)}, 0)}{\partial \rho_0} \cdot \rho_0^{(1)} + \frac{\partial F(\rho_0^{(0)}, 0)}{\partial h} + \omega^{(0)}(R_1^2 - R_0^2) + \int \kappa_{12} \rho_2^{(0)} d\tau_2, \tag{19b}$$

$$\begin{aligned} &\frac{\partial F(\rho_1^{(0)}, 0)}{\partial \rho_1} \cdot \rho_1^{(2)} + \frac{\partial^2 F(\rho_1^{(0)}, 0)}{\partial \rho_1^2} \cdot \rho_1^{(1)2} + 2 \frac{\partial^2 F(\rho_1^{(0)}, 0)}{\partial \rho_1 \partial h} \rho_1^{(1)} \\ &\quad + \frac{\partial^2 F(\rho_1^{(0)}, 0)}{\partial h^2} = \frac{\partial F(\rho_0^{(0)}, 0)}{\partial \rho_0} \rho_0^{(2)} + \frac{\partial^2 F(\rho_0^{(0)}, 0)}{\partial \rho_0^2} \cdot \rho_0^{(1)2} \\ &\quad + 2 \frac{\partial^2 F(\rho_0^{(0)}, 0)}{\partial \rho_0 \partial h} \rho_0^{(1)} + \frac{\partial^2 F(\rho_0^{(0)}, 0)}{\partial h^2} + 2\omega^{(1)}(R_1^2 - R_0^2) \\ &\quad + \int \kappa_{12} \rho_2^{(1)} d\tau_2. \end{aligned} \tag{19c}$$

Equation (19a) is satisfied by $\rho_1^{(0)} = \rho_0^{(0)}$, which makes ρ_1 equal ρ_0, as it should, when the point P_1 coincides with P. Since $\rho_0^{(0)}$ is a constant, $\rho_1^{(0)}$ is constant on Σ, satisfying one equilibrium condition. The center of gravity condition requires that the center of gravity of the mass $(\Sigma, \rho_0^{(0)})$, and hence of $(\Sigma, 1)$, lie on the axis A. Solving equation (19b) for $\rho_1^{(1)}$, we have a constant added to a constant multiple of

$$\omega^{(0)} R_1^2 + \int \kappa_{12} \rho_2^{(0)} d\tau_2.$$

* It is only in computing these derivatives that we need restrictions on the kind of function $F(\rho, h)$ may be. Any function which leads to equations (19a), (19b), (19c) on differentiating as to h, and for which the coefficient of $\rho_1^{(1)}$ and $\rho_1^{(2)}$ is not zero, may be used.

This expression is therefore constant on Σ under precisely the same conditions as the function $\rho_1^{(1)}$ of the preceding paragraphs, and differs from that function only by an added and a multiplied constant. Hence when Σ is an ellipsoid and A is a diameter of Σ, the function $\rho_1^{(1)}$ is either not constant on Σ or it is given by a formula similar to (17):

$$\rho_1^{(1)} = C_1 \mu + C_2, \qquad C_1 \neq 0.$$

When we consider $\rho_2^{(2)}$, as found from (19c), we see that, since $\rho_1^{(1)}$ is constant on Σ, $\rho_2^{(2)}$ is constant on Σ if and only if

$$2\omega^{(1)} R_1^2 + \int \kappa_{12} \rho_2^{(1)} d\tau_2$$

is constant on Σ. This leads at once, as before, to the conclusion $a = b = c$.

Hence we have

THEOREM 3. *Let Σ and A be as described in Theorem* 1. *Under the same hypotheses* (a), (b), *or the hypothesis of Theorem* 2, *it is impossible to determine ω, ρ_0, and ρ as power series in h so that equation* (5) *shall be satisfied, ρ shall be constant when $h = 0$, and $(\Sigma, \rho, \omega, A)$ shall be a system of equilibrium for all values of h near $h = 0$.*